Unimaths Intro WORKBOOK

An Introduction to the Foundations of First Year University Mathematics

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WORKBOOK

3rd Edition

UNIMATHS

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Welcome

You are one of those brave few who have decided to take on the challenge of first year university mathematics. Many students are ill prepared for such a great leap. Once at university there is a horrible snowball effect whereby not knowing what is happening in the work at the very beginning means that you do not know what is happening when the work becomes more in-depth. By then there isn't any time to try to pick up the beginning work. Very soon you are in freefall towards failure.

The good news is that this can easily be avoided. A small amount of preparation before the beginning can give you an edge that keeps you on top of your work from the beginning.

Enter *Unimaths Intro*. The work in these pages is easy going and relaxed. There is no pressure to learn or memorise any of this material for tests or exams. Each lesson is designed to give you an insight into a particular area of first year maths and then, just as it starts to become too involved, the lesson is over and you will have gained an idea of the basics of what to expect in that particular section. There are not a lot of exercises, just enough to give you a feel for working with the concepts that have been covered. Once you have tried the exercises you can look at the solutions at the back of each lesson to see how each exercise is done.

The purpose of *Unimaths Intro* is to introduce basic concepts, so the work is interesting and easy to read. It is the kind of workbook that you can quite comfortably fit into your grade 12 year schedule or you could use it as support during your first year at university. The lessons are compact so that you could even spend a day looking over a lesson before starting that section in your first year mathematics course.

In writing these lessons I have assumed that you have knowledge of grade 12 mathematics. However, the lessons on Convergence and Power Series and those on Integration are the only lessons which require you to have covered a section on sequences and series at school level. Feel free to skip these lessons until you have gone over that work in class. Some of the other sections will cover the same work you do in your grade 12 year but will look at it from a different perspective which will compliment your school syllabus.

In keeping with the theme of this workbook I am going to keep my welcome short but concise so it is here that I wish you all the best and I'll chat to you again at the end of the workbook. I do hope you enjoy these lessons, have fun.

Chris

Complex Numbers

Lesson 1: Introduction

Mathematics is a human invention. We invent ideas as we need them. Pretend for a moment that you live in a world that only has positive whole numbers, and it is good. Then someone comes along, a mathematics inventor (aka mathematician), and tells you about negative numbers, they ask you to:

"Take any number, say 9, and put a little line in front of it, -9."

Then they show you how to work with this new kind of number and they call it 'negative'. At first you would be very skeptical but as you learnt about it, you would realize how useful it could be. That is what is happening with imaginary numbers. Up until now you have lived in a world without them and it will take some getting used to, but soon you will know how to work with them and they will seem as normal as the negative whole numbers that you are familiar with.

Imaginary Numbers

To understand what a complex number is, we first have to understand what an imaginary number is. Remember that we cannot take the square root of a negative number. Example:

$$\sqrt{-2},\ \sqrt{-3}\ \text{ or }\ \sqrt{-4}\ \text{ are bad news}$$

Or that when we used $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ to find the roots of a quadratic, if b^2-4ac (the part under the $\sqrt{\ }$) was negative, then the roots of the quadratic did not exist.

Now, we want to stop running away from these situations and others like them. We start by removing the negative from under the $\sqrt{\ }$. Example:

$$\sqrt{-2} = \sqrt{2\times(-1)} = \sqrt{2}\sqrt{-1}$$

$$\sqrt{-3} = \sqrt{3\times(-1)} = \sqrt{3}\sqrt{-1}$$

$$\sqrt{-4} = \sqrt{4\times(-1)} = \sqrt{4}\sqrt{-1}$$

Then we make a bold move and declare $\sqrt{-1}$ an imaginary number and call it i so that:

$$\boxed{\sqrt{-1} = i}$$

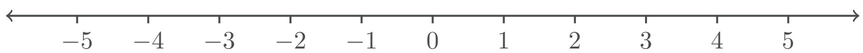
Therefore we can rewrite the surds from above like this:

$$\sqrt{-2} = \sqrt{2}i$$

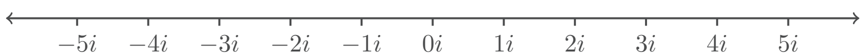
$$\sqrt{-3} = \sqrt{3}i$$

$$\sqrt{-4} = \sqrt{4}i = 2i$$

$\sqrt{2}i$, $\sqrt{3}i$ and $2i$ are all imaginary numbers. Remember how all real numbers exist on the real number line?



Well there is also an imaginary number line:



Every imaginary number lives on this imaginary number line. It should be obvious that:

$$0i = 0 \quad \text{and} \quad 1i = i$$

Great! So everything is sorted. Well not really. It's around this point that you may be thinking, "Wait a minute, how can we just make up a different number line?" The simple answer to that question is; because we needed to. Let me explain:

We have already seen that the set of natural numbers ($\mathbb{N}$) is not closed under subtraction. This means that if we subtract one natural number from another we might not get a natural number as our answer! Eg: $5 - 9 = -4$. Here we see that 5 is a natural number and 9 is also a natural number but the answer, -4 is not a natural number, its an integer. So, because ($\mathbb{N}$) is not closed, we need the integers ($\mathbb{Z}$). But the set of integers is not closed under division, which leads us to the rational numbers ($\mathbb{Q}$). The set of rationals is not closed under square roots (or, in general, exponentiation), leading us to the real numbers ($\mathbb{R}$). But it turns out that the set of reals is not closed under square root either, since we cannot take the square root of a negative number. In this chapter we look at a new set of numbers, called the complex numbers ($\mathbb{C}$), that is closed under exponentiation (recall that taking a square root is the same as raising to the power of $\frac{1}{2}$, hence exponentiation).

Question 1:

Simplify the following:

$\sqrt{-9}$	
$\sqrt{-\frac{25}{4}}$	
i^2	
$(-i)^2$	

Just think about the last two, they are actually quite easy.

Complex Numbers

Once we understand imaginary numbers, the idea of a complex number will come easily. A complex number is formed by adding a real number to an imaginary number: $5 + 3i$. Or subtracting, it doesn't really matter: $4 - 6i$.

With $5 + 3i$ the 5 is called the real part and the 3 is called the imaginary part and $5 + 3i$ is a number, not a sum, a number, a complex number. In fact, we can think of a real number as a complex number with imaginary part equal to 0 and an imaginary number as a complex number with real part equal to 0.

So

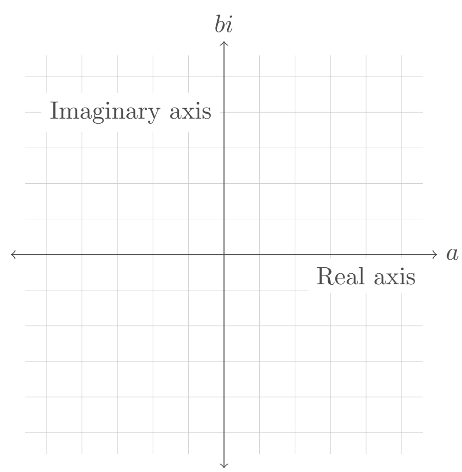
$$6 = 6 + 0i$$

and

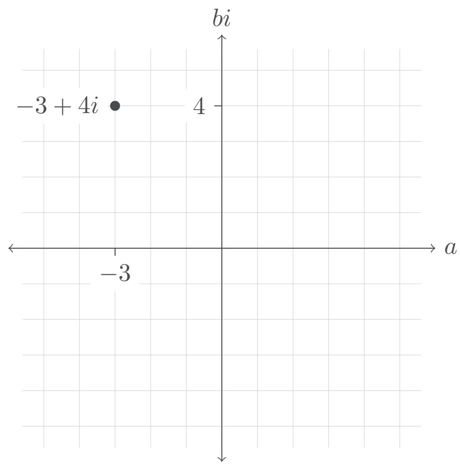
$$3i = 0 + 3i$$

In general, z and w are used to represent complex numbers, like x and y for the reals. Also, a represents the real part and b represents the imaginary part. So, in general, $z = a + bi$.

But which number line does a complex number live on? It can't go on either! It has a bit of each number line in it so what we do is use both number lines to form the axes of a plane. We do it like this:



We no longer think of a number line but rather a number plane called the **complex plane**. Any complex number can be represented as a point on this plane. Example:

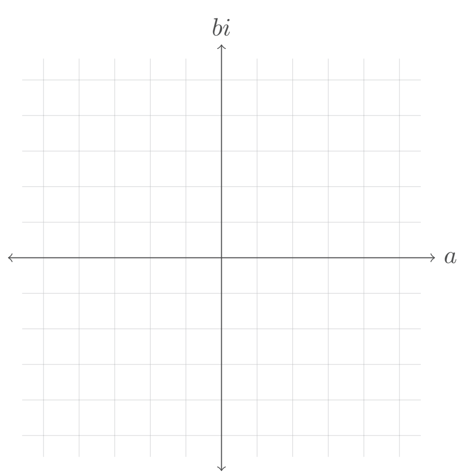


Note: the complex plane also contains all other types of numbers; real, rational, irrational, integer and natural. They are all on the real axis, where they have always been. The imaginary numbers all live on the imaginary axis, as we would expect.

Question 2:

Fill in the following complex numbers on the complex plane below:

- $2+3i$
- $2-3i$
- $1+i$
- $1-i$
- 1
- i
- 0



We will now look at 6 different ideas: addition, subtraction, multiplication, division, complex conjugation and modulus.

Addition/ Subtraction

We look at them together because they are really easy. To add complex numbers, we add their corresponding real and imaginary parts. Example:

$$(1 + 2i) + (2 + 3i) = 3 + 5i$$

The same applies to subtraction but we subtract their corresponding parts.

Question 3:

$(-5 + 2.5i) + (1 - 0.5i)$	
$(-5 + 2.5i) - (1 - 0.5i)$	
$(2.63) + (3.7i)$	
$(9 - i) - (i)$	

Multiplication

Multiplying requires a bit more thought but is also quite easy. It's just like multiplying out brackets. Example:

$$\begin{aligned} &(2 + i)(3 + 2i) \\ =&2.3 + 2.2i + 3.i + 2i.i \\ =&6 + 4i + 3i - 2 \\ =&4 + 7i \end{aligned}$$

Question 4:

$(4)(1 + i)$	
$(5 + i)(5 + i)$	
$(3)(3i)$	
$(0)(5.62 - 3.71i)$	
$(2 + i)(2 - i)$	

Complex conjugate

Before we look at division we first have to understand what a complex conjugate is. What did you notice about the last exercise in question 4? When those two complex numbers were multiplied, we got a real number. That is because (2 + i) and (2 - i) are complex conjugates. The complex conjugate of a complex number is another complex number which is the same in every way except that the sign of its imaginary part is changed.

Question 5:

Find the complex conjugates of the following complex numbers:

$3+i$	
$3-i$	
$-3+i$	
$-3-i$	
i	
3	
0	

Whenever we multiply a complex number by its complex conjugate, the result is a real number. You should test your answers for question 5 to see that this holds. If we are told that z is a complex number then we write its conjugate as $\overline{z}$. So, if $z = 2+3i$, then $\overline{z} = 2-3i$.

Division

What if we get given something like this: $\frac{5+i}{4-3i}$, how do we work with this to get it into a complex number with separate real and imaginary parts? We could try splitting up the fraction. Example:

$$\frac{5}{4-3i} + \frac{i}{4-3i}$$

But which part is the real part and which is the imaginary part? Neither! Now we consider a different situation; if the denominator were a purely real number, would splitting the fraction work? Example:

$$\frac{7+i}{2} = \frac{7}{2} + \frac{i}{2}$$

Is this a complex number? Yes! Why? Because it has separate real and imaginary parts. So if we go back to $\frac{5+i}{4-3i}$, the question becomes, how can we turn the denominator into a purely real number? Think about it before reading further.

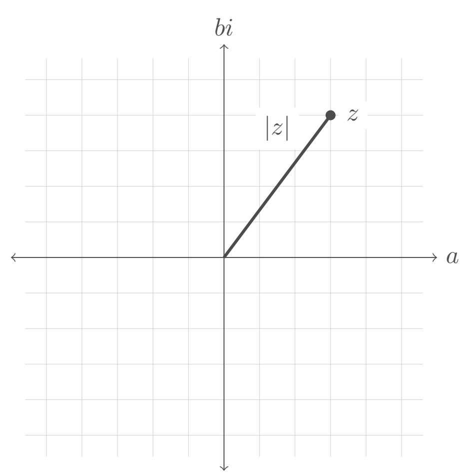
Complex conjugate to the rescue! If we multiply the denominator by its complex conjugate, we get a real number. Then we can separate the numerator into real and imaginary parts. So, we multiply the numerator and the denominator by $4+3i$. Try this for yourself. When you are done, you should have a complex number.

Question 6:

$\frac{4}{1+i}$	
$\frac{1+i}{4}$	
$\frac{7-9i}{0}$	
$\frac{0}{7-9i}$	
$\frac{3-2i}{2-3i}$	
$\frac{1}{i}$	

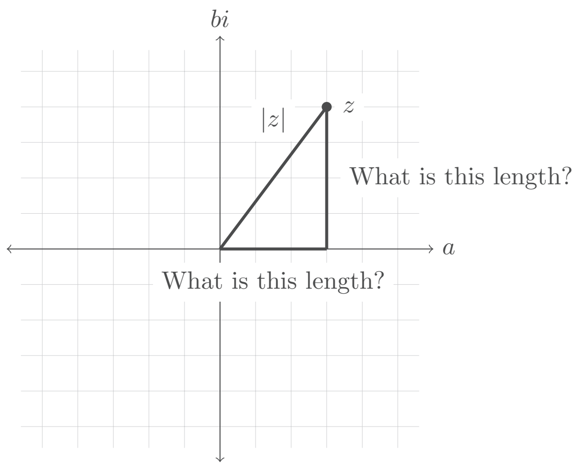
Modulus

The modulus of a complex number is simply the distance between it and the origin of the complex plane.



So $|z|$ is the distance from 0 (that is, from the origin) to z on the complex plane. We can also see that this is the same as the absolute value for real numbers. If a number is on the real axis, the distance from 0 to that number is the same as the absolute value of that number.

Rather than measuring this type of distance directly, using a ruler, we have already learned a method for calculating it. For example, if $z = 3 + 4i$:



Now how would we find $|z|$?

We know that the vertical length is 4 and the horizontal length is 3 so we can use the theorem of Pythagoras to find $|z|$. Example:

$$\begin{aligned} |z|^2 &= 3^2 + 4^2 \\ \therefore |z| &= \sqrt{3^2 + 4^2} \\ &= 5 \end{aligned}$$

But what happened to the i? It is important to remember that we are only using the magnitude of the real and imaginary parts, when using the theorem of Pythagoras to find $|z|$.

Question 7:

Give a general expression for $|z|$. What that means is if $z = a + bi$, what is $|z|$ in terms of a and b? (Hint: Use the theorem of Pythagoras.)

Question 8:

Find the modulus for each of the following:

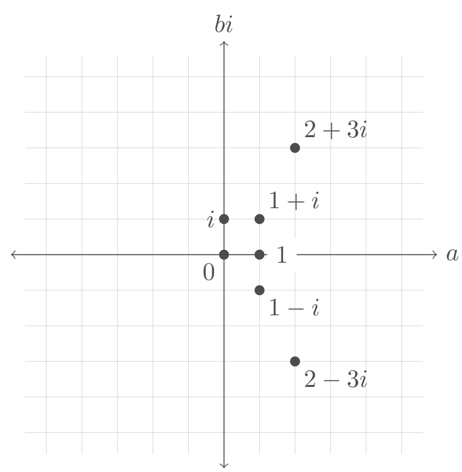
$-5+6i$	
$1+i$	
$-1+i$	
$1-i$	
$-1-i$	
3	
$3i$	
0	

Solutions

Solution 1:

$\sqrt{-9}$	$=3\sqrt{-1}$ $=3i$
$\sqrt{-\frac{25}{4}}$	$=\frac{5}{2}\sqrt{-1}$ $=\frac{5}{2}i$
i^2	$=i.i$ $=\sqrt{-1}.\sqrt{-1}$ $=-1$
$(-i)^2$	$=(-i)(-i)$ $=i.i$ $=\sqrt{-1}.\sqrt{-1}$ $=-1$

Solution 2:



Solution 3:

$(-5+2.5i)+(1-0.5i)$	$-4+2i$
$(-5+2.5i)-(1-0.5i)$	$-6+3i$
$(2.63)+(3.7i)$	$2.63+3.7i$
$(9-i)-(i)$	$9-2i$

Solution 4:

$(4)(1+i)$	$4+4i$
$(5+i)(5+i)$	$=(25+5i+5i-1)$ $=24+10i$
$(3)(3i)$	$9i$
$(0)(5.62-3.71i)$	$=(0+0i)$ $=0$
$(2+i)(2-i)$	$=(4+2i-2i+1)$ $=5$

Solution 5:

$3+i$	$3-i$
$3-i$	$3+i$
$-3+i$	$-3-i$
$-3-i$	$-3+i$
i	$-i$
3	3
0	0

Solution 6:

$\frac{4}{1+i}$	$=\frac{4}{1+i} \times \frac{1-i}{1-i}$ $=\frac{4-4i}{(1+i)(1-i)}$ $=\frac{4-4i}{1+i-i+1}$ $=\frac{4-4i}{2}$ $=\frac{4}{2} - \frac{4i}{2}$ $=2-2i$
$\frac{1+i}{4}$	$=\frac{1}{4} + \frac{i}{4}$ $=\frac{1}{4} + \frac{1}{4}i$
$\frac{7-9i}{0}$	Undefined
$\frac{0}{7-9i}$	$=\frac{0}{7} - \frac{0}{9i}$ $=0$
$\frac{3-2i}{2-3i}$	$=\frac{3-2i}{2-3i} \times \frac{2+3i}{2+3i}$ $=\frac{6-4i+9i+6}{4-6i+6i+9}$ $=\frac{12+5i}{13}$ $=\frac{12}{13} + \frac{5}{13}i$
$\frac{1}{i}$	$=\frac{1}{i} \times \frac{-i}{-i}$ $=\frac{-i}{1}$ $=-i$

Solution 7:

If:

$$z = a + bi$$

Then:

$$|z|^2 = a^2 + b^2$$

Therefore:

$$|z| = \sqrt{a^2 + b^2}$$

Solution 8:

Find the modulus for each of the following:

$-5+6i$	$\|-5+6i\|$ $=\sqrt{(-5)^2+6^2}$ $=\sqrt{61}$
$1+i$	$\|1+i\|$ $=\sqrt{1^2+1^2}$ $=\sqrt{2}$
$-1+i$	$\|-1+i\|$ $=\sqrt{(-1)^2+1^2}$ $=\sqrt{2}$
$1-i$	$\|1-i\|$ $=\sqrt{1^2+(-1)^2}$ $=\sqrt{2}$
$-1-i$	$\|-1-i\|$ $=\sqrt{(-1)^2+(-1)^2}$ $=\sqrt{2}$
3	$\|3\|$ $=\sqrt{3^2+0^2}$ $=3$
$3i$	$\|3i\|$ $=\sqrt{0^2+3^2}$ $=3$
0	$\|0\|$ $=\sqrt{0^2+0^2}$ $=0$

Complex Numbers

Lesson 2: A different way to write complex numbers

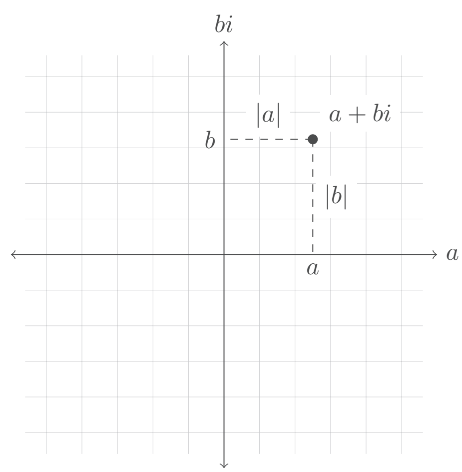
Mod-arg form

Let's, for a moment, think back to scientific notation. Here's a reminder of what scientific notion looks like:

$$300\ 000\ 000 = 3 \times 10^8$$

Why do we use scientific notation? Because it is easier to work with big numbers in this form. Note that 300 000 000 and 3×10^8 are the exact same thing but written in different ways.

In this lesson we are going to make the complex numbers look different so that they are easier to work with. Here's how: We know that every complex number, z, is written in the form $a + bi$. I have put $a + bi$ on the complex plane below (I could have put it anywhere because it is a variable complex number).

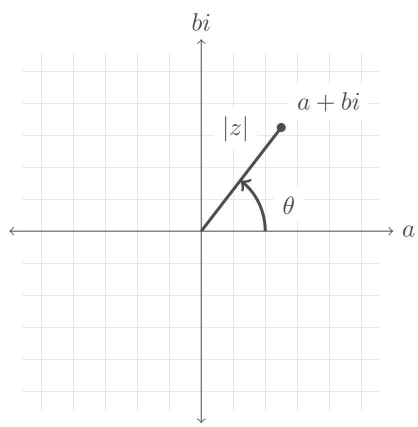


a is its position along the real axis and b is its position along the imaginary axis. We also know that $|a|$ is the distance between $a+bi$ and the imaginary axis and $|b|$ is the distance between $a+bi$ and the real axis. Let's get a feel for this:

Question 1:

	Distance from the imaginary axis	Distance from the real axis
$2+3i$		
$-4+i$		
$-5-2i$		
$2-3i$		

Here comes the key step. Instead of giving you a and b to describe a complex number, I give you $|z|$ and θ, where θ represents the angle between $|z|$ and the real axis. Then I ask you to use this new information to find out what a and b are equal to.



Question 2:

Rewrite a and b in terms of $|z|$ and θ. In other words, if you wanted to write a and b but all you had were $|z|$ and θ , how would you write a and b using $|z|$ and θ ? (Hint: Use trigonometry.)

$a =$

$b =$

$a + bi =$

Simplify this expression by factorization.

Question 3:

Given the following values of $|z|$ and θ, each pair describing a complex number, write each complex number in its standard form. The standard form is $a + bi$.

$\|z\|$	θ	a	b	$a+bi$
$\sqrt{2}$	$45°$			
$\sqrt{2}$	$135°$			
2	$-30°$			
1	$90°$			
3	$180°$			
3	$0°$			
$\sqrt{2}$	$405°$			

We know that $|z|$ is called the modulus and θ is called the argument. When we write a complex number using $|z|$ and θ we say that it is in mod-arg form.

Question 4:

What do you notice about the first and the last exercises in question 3? Can you think of a reason for this?

Question 5:

Now go in the opposite direction. Rewrite $|z|$ and θ in terms of a and b. In other words, if you wanted to write $|z|$ and θ but all you had were a and b, how would you use a and b to get $|z|$ and θ? (Hint: Use the theorem of Pythagoras for $|z|$ and trigonometry for θ.)

$$|z| =$$

$$\theta =$$

Question 6:

Rewrite the following complex numbers in mod-arg form.

$a + bi$	$\|z\|$	θ	$\|z\|(\cos\theta + i\sin\theta)$
$3 + 4i$			
$-3 + 4i$			
$-3 - 4i$			
$3 - 4i$			
$1 - \sqrt{3}i$			
$-1 - i$			
-5			
$-2i$			
6			
i			

We can see that every complex number can be written in terms of a and b OR $|z|$ and θ. Still the same old complex number only written in a different way. Next we will look at why we sometimes want to write complex numbers in the mod-arg form rather than the standard form.

Mod-arg multiplication

We have seen that if we are given the modulus and argument of a complex number we can write our complex number in the following way:

$$z = |z|(\cos\theta + i\sin\theta)$$

Mod-arg form is very useful for multiplication. Here is the general rule for how to multiply two complex numbers that are in mod-arg form

Let

$$z = |z|(\cos\theta + i\sin\theta)$$
$$w = |w|(\cos\gamma + i\sin\gamma)$$

then

$$z \cdot w = |z||w|(\cos(\theta+\gamma) + i\sin(\theta+\gamma))$$

So we multiply the modulii (plural of modulus) of the two complex numbers and add the arguments.

If we wanted to multiply $5(\cos 30^\circ + i\sin 30^\circ)$ by $2(\cos 15^\circ + i\sin 15^\circ)$ we would get $10(\cos 45^\circ + i\sin 45^\circ)$.

See how much easier multiplication is in mod-arg form! If you would like to know why this method works, take a look at the proof below:

Mod-arg multiplication proof:

Let

$$z = |z|(\cos\theta + i\sin\theta) \qquad \text{and} \qquad w = |w|(\cos\gamma + i\sin\gamma)$$

$$\begin{aligned}
z \cdot w &= [|z|(\cos\theta + i\sin\theta)][|w|(\cos\gamma + i\sin\gamma)] \\
&= |z||w|(\cos\theta + i\sin\theta)(\cos\gamma + i\sin\gamma) \\
&= |z||w|[\cos\theta\cos\gamma + i\sin\gamma\cos\theta + i\sin\theta\cos\gamma - \sin\theta\sin\gamma] \\
&= |z||w|[(\cos\theta\cos\gamma - \sin\theta\sin\gamma) + i(\sin\gamma\cos\theta + \sin\theta\cos\gamma)] \\
&= |z||w|(\cos(\theta+\gamma) + i\sin(\theta+\gamma))
\end{aligned}$$

The last step makes use of the trig compound angle formula.

Question 7:

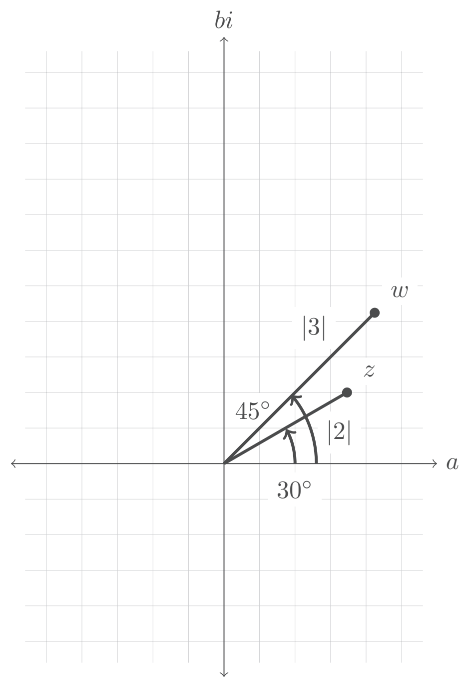
Multiply the following complex numbers:

$$[3(\cos 16^\circ + i\sin 16^\circ)][2(\cos 9^\circ + i\sin 9^\circ)] =$$

$$[5(\cos 101^\circ + i\sin 101^\circ)][4(\cos 200^\circ + i\sin 200^\circ)] =$$

Question 8:

Without doing any working, draw $z \cdot w$ onto the complex plane:



Question 9:

Rewrite the following two complex numbers in mod-arg form, multiply them together and convert the answer back into standard form:

$$(1 + \sqrt{3}i) \qquad (\sqrt{3} + i)$$

Mod-arg form of $(1 + \sqrt{3}i)$:

Mod-arg form of $(\sqrt{3} + i)$:

Product in mod-arg form:

Product in standard form:

Solutions

Solution 1:

	Distance from the imaginary axis	Distance from the real axis
$2+3i$	2	3
$-4+i$	4	1
$-5-2i$	5	2
$2-3i$	2	3

Solution 2:

$$\begin{aligned} a &= |z|\cos\theta \\ b &= |z|\sin\theta \\ a+bi &= |z|\cos\theta + i|z|\sin\theta \qquad \text{(The } i \text{ goes in front of the } |z|\sin\theta\text{.)} \\ &= |z|(\cos\theta + i\sin\theta) \end{aligned}$$

Solution 3:

$\lvert z\rvert$	θ	a	b	$a+bi$
$\sqrt{2}$	$45°$	1	1	$1+i$
$\sqrt{2}$	$135°$	-1	1	$-1+i$
2	$-30°$	$\sqrt{3}$	-1	$\sqrt{3}-i$
1	$90°$	0	1	i
3	$180°$	-3	0	-3
3	$0°$	3	0	3
$\sqrt{2}$	$405°$	1	1	$1+i$

Solution 4:

Even though the argument (θ) is different in each exercise the $a+bi$ form of the complex number is the same. The reason for this is that $405°$ is $360°$ greater than $45°$, which means $405°$ is exactly one full revolution more than $45°$.

Solution 5:

$$|z| = \sqrt{a^2+b^2}$$
$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

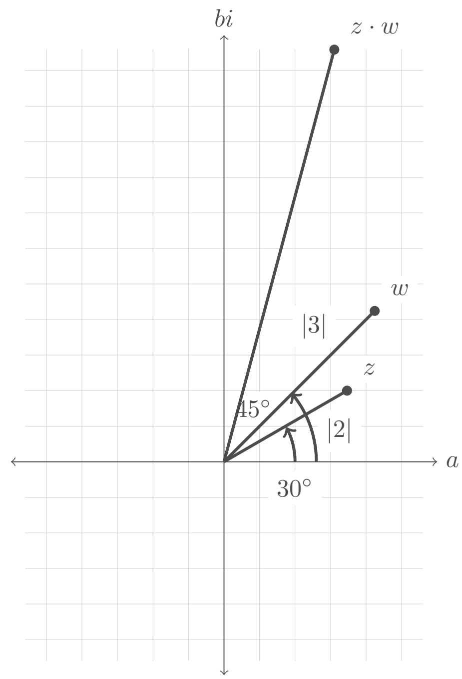
Solution 6:

$a+bi$	$\lvert z\rvert$	θ	$\lvert z\rvert(\cos\theta + i\sin\theta)$
$3+4i$	5	$53.13°$	$5(\cos 53.13° + i\sin 53.13°)$
$-3+4i$	5	$126.87°$	$5(\cos 126.87° + i\sin 126.87°)$
$-3-4i$	5	$233.13°$	$5(\cos 233.13° + i\sin 233.13°)$
$3-4i$	5	$-53.13°$	$5(\cos(-53.13°) + i\sin(-53.13°))$
$1-\sqrt{3}i$	2	$-60°$	$2(\cos(-60°) + i\sin(-60°))$
$-1-i$	$\sqrt{2}$	$225°$	$\sqrt{2}(\cos 225° + i\sin 225°)$
-5	5	$180°$	$5(\cos 180° + i\sin 180°)$
$-2i$	2	$270°$	$2(\cos 270° + i\sin 270°)$
6	6	$0°$	$6(\cos 0° + i\sin 0°)$
i	1	$90°$	$(\cos 90° + i\sin 90°)$

Solution 7:

$$[3(\cos 16° + i\sin 16°)][2(\cos 9° + i\sin 9°)] = 6(\cos 25° + i\sin 25°)$$
$$[5(\cos 101° + i\sin 101°)][4(\cos 200° + i\sin 200°)] = 20(\cos 301° + i\sin 301°)$$

Solution 8:



$z \cdot w$ will have a length (modulus) of 6 and an angle (argument) of $75°$

Solution 9:

Mod-arg form of $(1 + \sqrt{3}i)$: $2(\cos 60° + i \sin 60°)$

Mod-arg form of $(\sqrt{3} + i)$: $2(\cos 30° + i \sin 30°)$

Product in mod-arg form: $4(\cos 90° + i \sin 90°)$

Product in standard form:

$$\begin{aligned} a &= 4 \cos 90° \\ &= 4.0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} b &= 4 \sin 90° \\ &= 4.1 \\ &= 4 \end{aligned}$$

$$a + bi = 4i$$

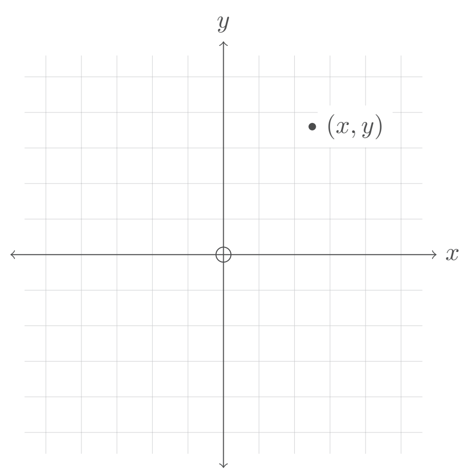
3D Vectors

Lesson 1: Introduction to 2D vectors

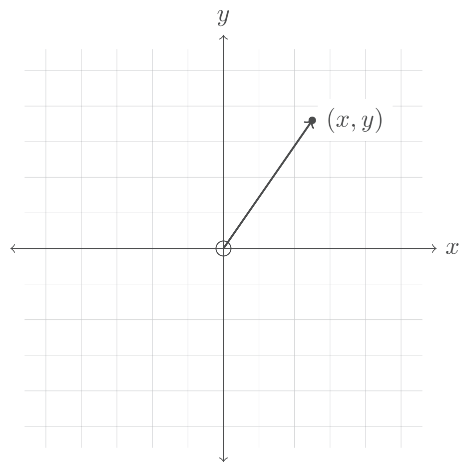
2D Vectors

At school you learnt about the Cartesian plane and vectors in 2 dimensions. Here we want to add a dimension and go up to 3 dimensions. This is useful since we perceive physical phenomena to unfold in 3-dimensional space. However, before we dive into 3D, we will spend this lesson looking at a few new concepts in 2D and then extend those concepts to 3D in the next lesson.

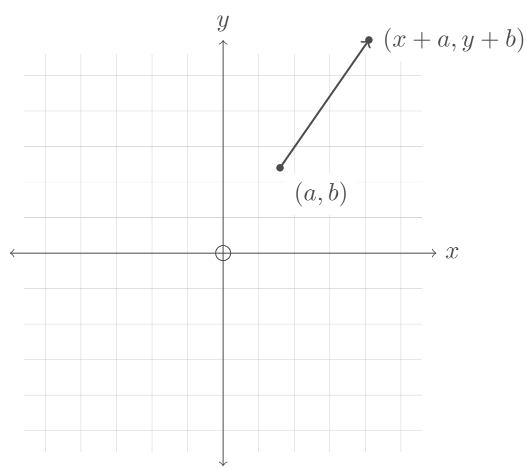
You are very comfortable with the 2D plane. You call it the Cartesian plane and you can represent any point on this plane as an ordered pair of numbers (x, y).



You may be familiar with the concept of a vector. If not, don't worry, it's a simple concept and you will learn about it here. The following ideas may seem simple, but they will build up to an important idea. Firstly, this is a vector in 2D:



It's just a line starting at any position on the Cartesian plane, going to whichever point we want it to go to. I have drawn mine starting at the origin because I wanted to. If you don't want it to start at the origin then you have to say where you want it to start from. If you wanted to draw the vector starting at (a, b) then you can do it like this:

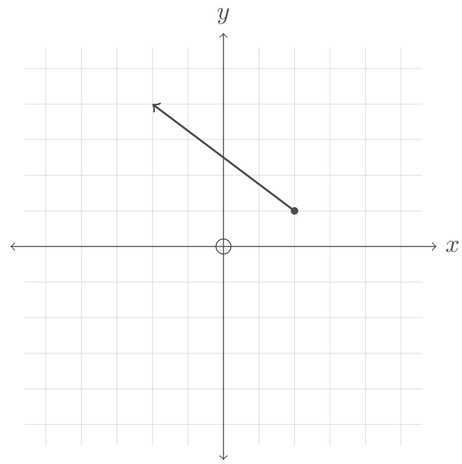


Vectors also have direction. That is why we draw an arrowhead at the end of the line. Think of vectors as geometry on a coordinate system. It's all about lines and points, that's it.

The two vectors in the above diagrams are exactly the same. The second vector is just the first vector starting at (a, b). The second vector ends at $(x + a, y + b)$ because its displacment along the 2 dimensions is x and y.

The vector in the first diagram starts at the origin and goes to point (x, y) so we call this vector $< x, y >$. The angle brackets around the x and y mean that we are talking about the vector that starts at the origin and goes to (x, y). The vector in the second diagram is also called $< x, y >$ but we must also state that it is starting at the point (a, b). So it doesn't matter where the vector is on the plane, we still describe it using the point that it would go to if it were starting at the origin. Example:

This is the vector $< -4, 3 >$ starting at the point $(2, 1)$



We call the x and y parts of a vector its components. If I give you the vector $< 5, 2 >$, then its x-component is 5 and its y-component is 2.

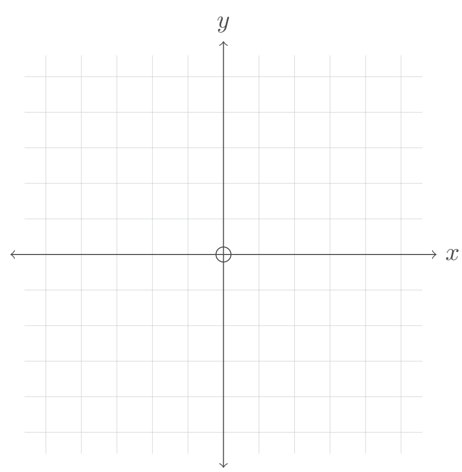
So what is the point of a vector? We have seen that a vector can point to a point! We will also see that vectors can be used to describe a line by pointing out all the points that make up the line, and this will become very useful in 3D. First we need to know how to use them and then we need to start thinking of them in 3D! Don't worry, we are still in 2D at the moment.

Vector addition

To add two vectors you simply add their corresponding components.

Question 1:

Sketch $< 1, 2 >$ and $< 3, 2 >$ on the set of axes:



Question 2:

Now add the two vectors together and sketch the answer on the same set of axes.

Question 3:

Now re-sketch $< 1, 2 >$ on the same set of axes but don't start at the origin, start at the end of $< 3, 2 >$ (in other words think of the end of $< 3, 2 >$ as the origin and then sketch $< 1, 2 >$ at this 'new origin').

Question 4:

What do you notice?

Question 5:

Now draw $< 3, 2 >$ starting at the end of $< 1, 2 >$, on the same set of axes.

Same result! So adding components of vectors and then sketching the answer OR placing them end to end gets you to the same point.

Question 6:

Add the following vectors: $< 4, 5 >$ and $< -1, 2 >$

Length of a vector

Vectors also have length. How do we find this length? Well isn't it just the same as finding the distance from the origin to a point? Yes, it is, and for that we need the theorem of Pythagoras. We put absolute value signs around a vector to show that we want to find its length. Example:

If I wanted to ask you to find the length of $< 2, -5 >$ I would ask you to find $| < 2, -5 > |$.

Example:

$$| < 2, -5 > | = \sqrt{2^2 + (-5)^2} = \sqrt{29}$$

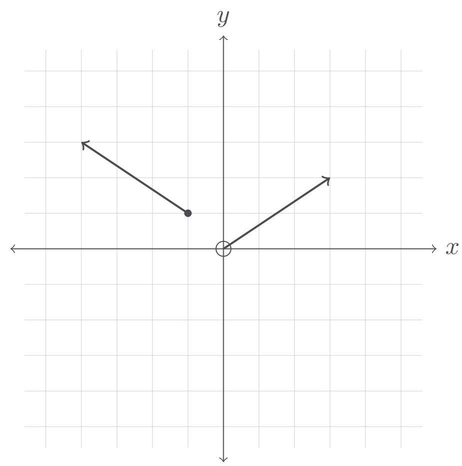
Question 7:

Compute the following lengths:

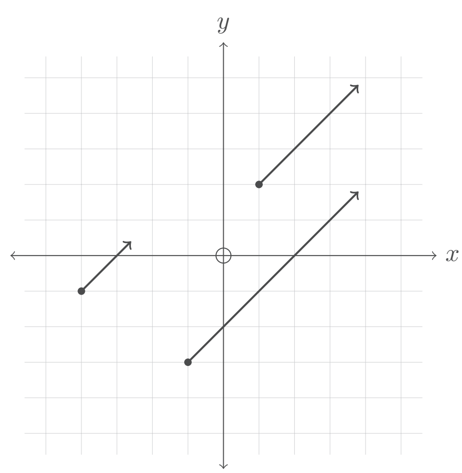
$\lvert < 1, 1 > \rvert$	
$\lvert < 2, 6 > \rvert$	
$\lvert < -2, 6 > \rvert$	
$\lvert < 2, -6 > \rvert$	

Direction of a vector

It should also be clear that vectors have direction. These two vectors have the same length but different directions:



These three vectors have different lengths but the same direction:



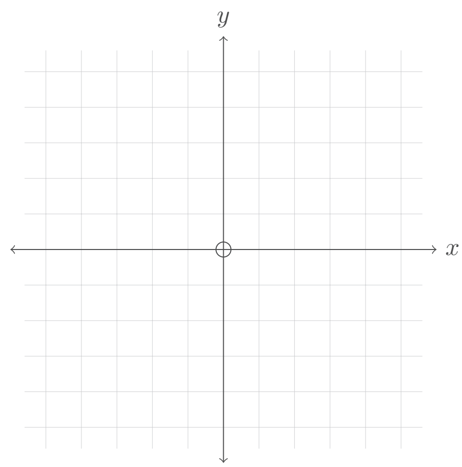
Subtraction

To do this, we need only subtract corresponding components. If we subtract one vector from another then the length and direction of the resultant vector is the same as the length and direction of the distance between the ends of the two vectors.

Question 8:

Sketch the following two vectors on the set of axes then subtract one from the other (don't worry about order) and sketch the answer on the same set of axes.

$$< 2, 3 > \text{ and } < 3, 2 >$$



Compare the resultant vector with the distance between the ends of the original two vectors. If we could pick it up and place it between the ends of the two original two vectors, it would fit perfectly!

If we subtract two vectors from one another then the length of the resultant vector is equal to the distance between the original two vectors.

Question 9:

Find the distance between $< -3, 2 >$ and $< 4, 4 >$

Scalar multiplication

In other words, number multiplication. Scalar is just another word for real number and all we are doing is multiplying the components of a vector by a real number. Example:

$$5 \cdot < 1, 2 >$$
$$= < 5, 10 >$$

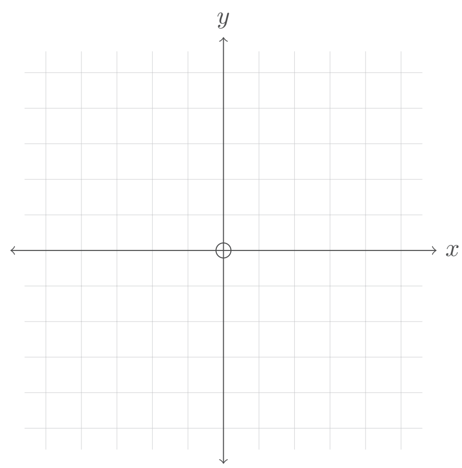
Question 10:

Can you think of what effect this has on a vector?

Question 11:

Multiply the following vectors by the given scalar and sketch them on the set of axes below.

	Resultant vector
$2\cdot <1,1>$	
$3\cdot <1,1>$	
$-2\cdot <1,2>$	



And now it all comes together

Let's put these vectors to use!

Vectors give us another way to describe a line. Instead of using an equation to describe all the points that make up a line we could use a collection of vectors to point to all the points on our line. We know that a line has an infinite number of points so we need an infinite number of vectors to point to all the points. To understand this concept let's start by thinking of a 'variable scalar vector'. This is a vector that is multiplied by a scalar that is actually a variable. If we call this variable λ (lambda) then our 'variable scalar vector' will look like this for example:

$$\lambda <5,2> \qquad \text{(where } \lambda \text{ can be any real number)}$$

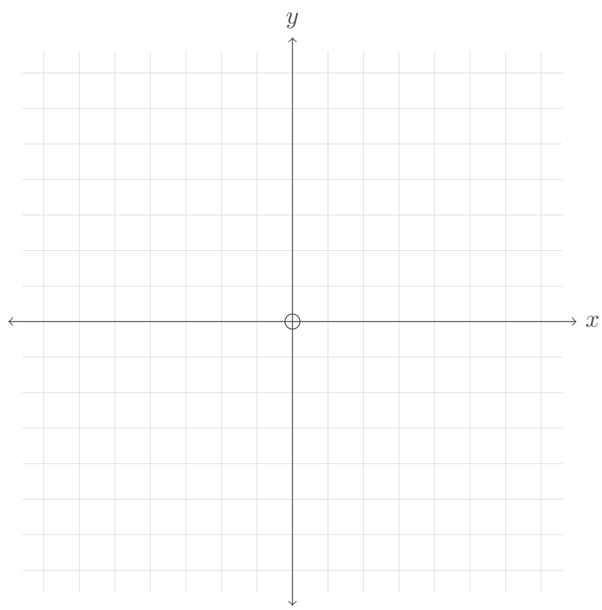
But how do we sketch a variable scalar vector? Well we would have to sketch the points at the ends of all possible vectors that $\lambda <5,2>$ represents. This may seem like a task that could take forever but try this question and you will see that it is not as difficult as you may think:

Question 12:

You are given the following variable scalar vector: $\lambda < 3, 2 >$

On the set of axes below sketch lightly in pencil, $\lambda < 3, 2 >$, given the following values for λ:

	Resultant vector
$\lambda = 1$	
$\lambda = -1$	
$\lambda = -2$	
$\lambda = 2$	

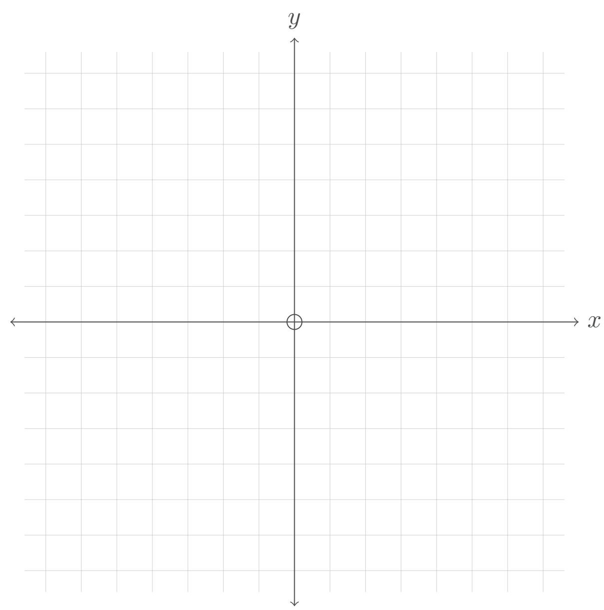


Draw a point, in pen, at the end of each vector. Can you see what is happening? By choosing only a few values for λ and sketching only a few of the points that $\lambda < 3, 2 >$ points to, we can already see that a straight line is forming. At this stage you can draw a line through those few points and you will have the line that $\lambda < 3, 2 >$ points out when λ is taken to be any real value. This is the same idea behind sketching a straight line of the form $y = mx + c$; you only have to find two points that satisfy the equation and that is enough the sketch the whole graph by drawing a line through those two points.

Question 13:

Sketch the following variable scalar vectors on the set of axes below:

$$\lambda < 1,1 >$$
$$\lambda < 5,-1 >$$
$$\lambda < 0,3 >$$
$$\lambda < 2,2 >$$



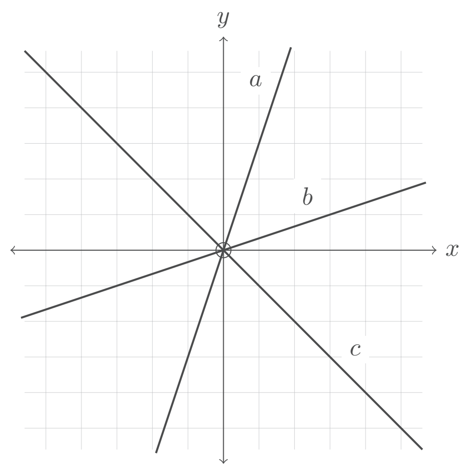
A 'variable scalar vector' is just a line through the origin and we call it a vector equation.

Question 14:

What did you notice about the first and last vector equation above and why?

Question 15:

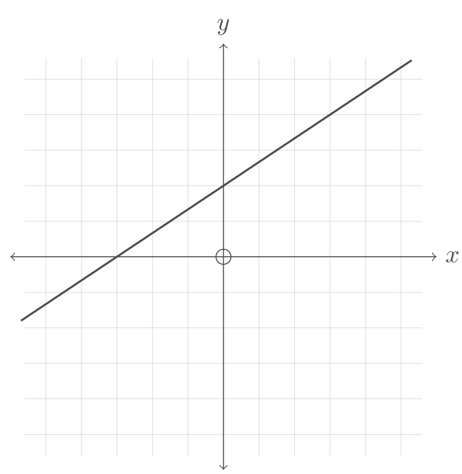
Now work backwards. Find vector equations for the following lines: (No working out is required.)



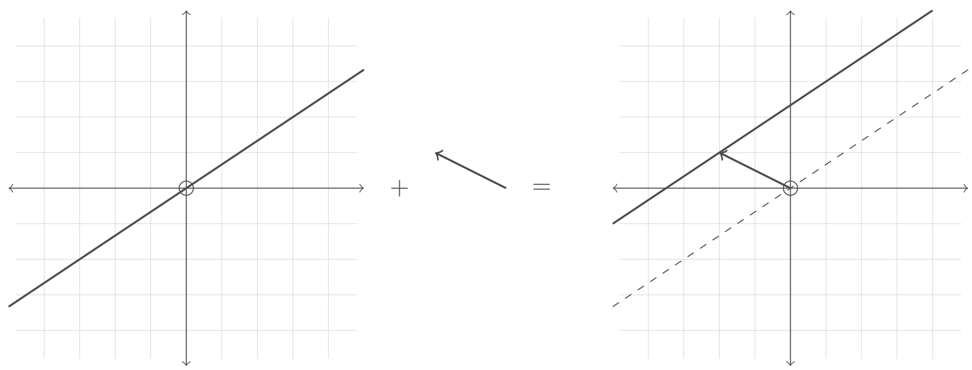
Line	Vector equation
a	
b	
c	

Lines that don't run through the origin

But what if we want a line that doesn't run through the origin? Example:



How do we use vectors to write a vector equation for this line? To understand how to draw this line using vectors you have to think about how vector addition works. If we want to use the ends of a whole lot of vectors to point out a line that doesn't run through the origin then we need all those vectors to start at the end of some other vector! As though the end of the other vector forms a new origin somewhere out on the plane. Here is a picture to explain:



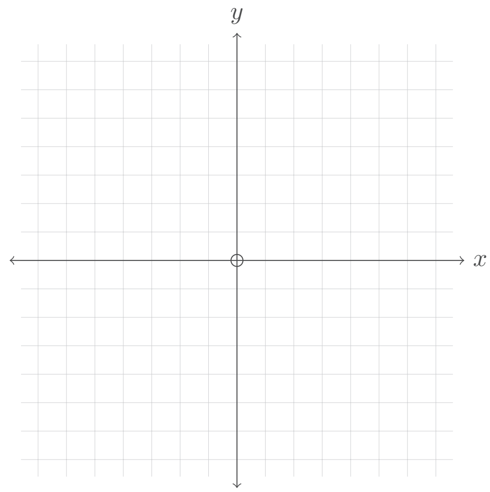
Variable scalar vector through origin | Add vector | Variable scalar vector not through origin

In a sense, we are using one vector ($\langle -2, 1\rangle$ in the example above) to move the origin of a variable scalar vector from $(0, 0)$ to somewhere else (in this example, the position $(-2, 1)$).

Question 16:

Plot the vectors below on the set of axes that follows and draw the line that they make with their ends (This line will represent the variable scalar vector $\lambda < 1, -1 >$): (Note: These vectors are still the kind that define a line that passes through the origin. I am asking you to draw them first because they lead on to drawing a line that doesn't pass through the origin.)

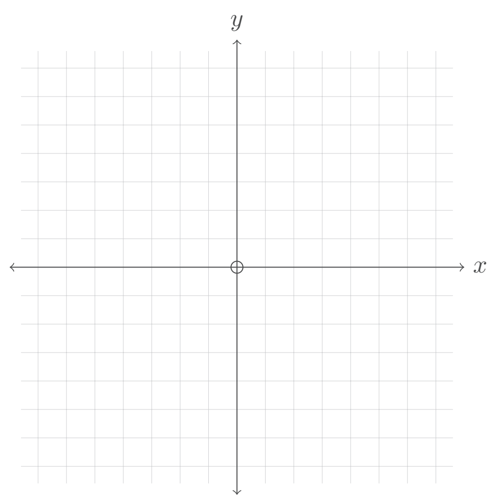
	Resultant vector
$1. < 1, -1 >$	
$2. < 1, -1 >$	
$-2. < 1, -1 >$	



Question 17:

Now plot the following vectors on the set of axes provided, lightly in pencil, and draw the line that they make with their ends:

$$1. < 1, -1 > + < 4, 4 >$$
$$2. < 1, -1 > + < 4, 4 >$$
$$-2. < 1, -1 > + < 4, 4 >$$



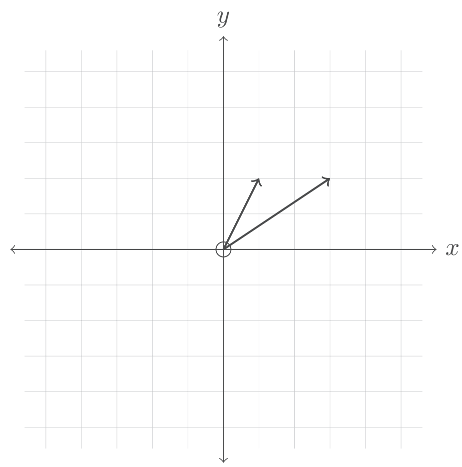
In general the vector equation for the line you have plotted in Question 17 is:

$$(x, y) = \lambda < 1, -1 > + < 4, 4 >$$

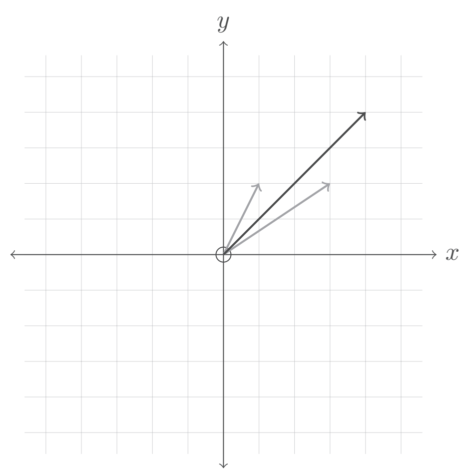
This is an example of a line that doesn't run through the origin.

Solutions

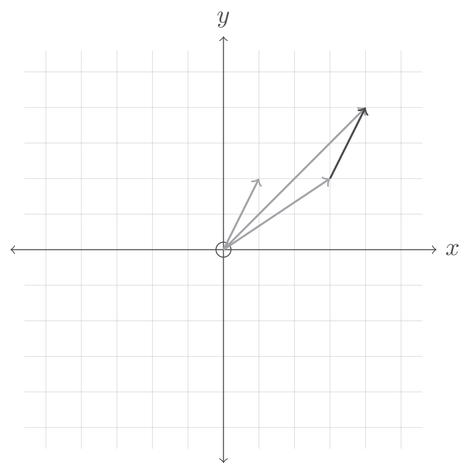
Solution 1:



Solution 2:



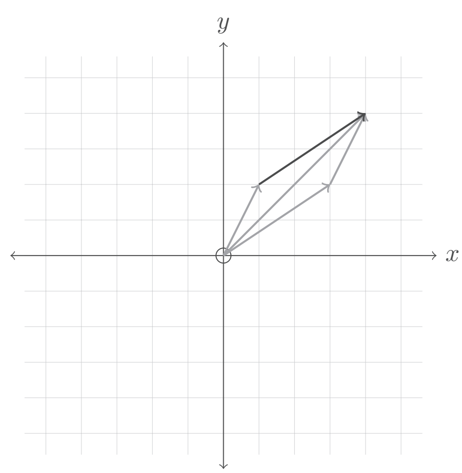
Solution 3:



Solution 4:

The vector that we get by adding $< 1, 2 >$ and $< 3, 2 >$ together points to the same point that the vector $< 1, 2 >$ points to when it is placed at the end of vector $< 3, 2 >$.

Solution 5:



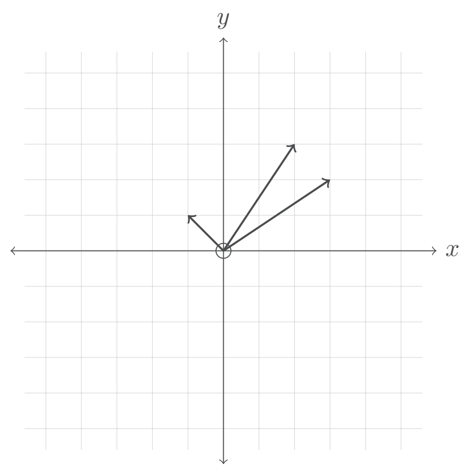
Solution 6:

$$< 4, 5 > + < -1, 2 > = < 3, 7 >$$

Solution 7:

$\lvert < 1, 1 > \rvert$	$= \sqrt{1^2 + 1^2}$ $= \sqrt{2}$
$\lvert < 2, 6 > \rvert$	$= \sqrt{2^2 + 6^2}$ $= \sqrt{40}$
$\lvert < -2, 6 > \rvert$	$= \sqrt{(-2)^2 + 6^2}$ $= \sqrt{40}$
$\lvert < 2, -6 > \rvert$	$= \sqrt{2^2 + (-6)^2}$ $= \sqrt{40}$

Solution 8:



Solution 9:

Find the difference:

$$< -3, 2 > - < 4, 4 > = < -7, -2 >$$

Find the length of this new vector:

$$\begin{aligned} \lvert < -7, -2 > \rvert &= \sqrt{(-7)^2 + (-2)^2} \\ &= \sqrt{53} \end{aligned}$$

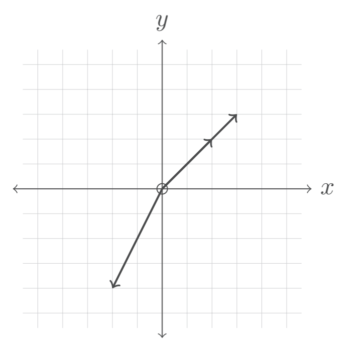
$\sqrt{53}$ is the distance between the two vectors.

Solution 10:

It changes the length of the vector.

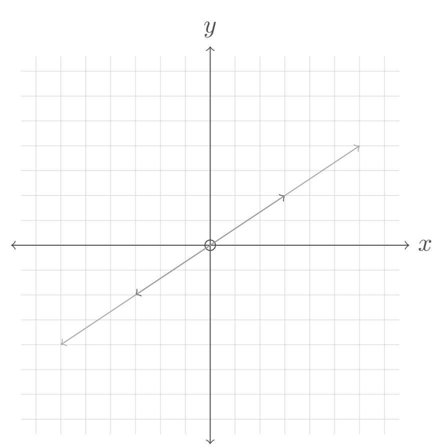
Solution 11:

	Resultant vector
$2\cdot <1,1>$	$<2,2>$
$3\cdot <1,1>$	$<3,3>$
$-2\cdot <1,2>$	$<-2,-4>$

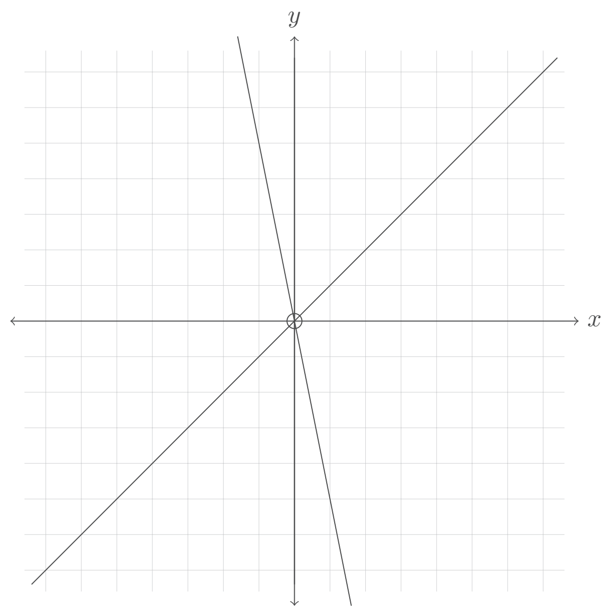


Solution 12:

	Resultant vector
$\lambda = 1$	$<3,2>$
$\lambda = -1$	$<-3,-2>$
$\lambda = -2$	$<-6,-4>$
$\lambda = 2$	$<6,4>$



Solution 13:



Note: $\lambda < 0, 3 >$ is on the y-axis and that is why we do not see it in this solution.

Solution 14:

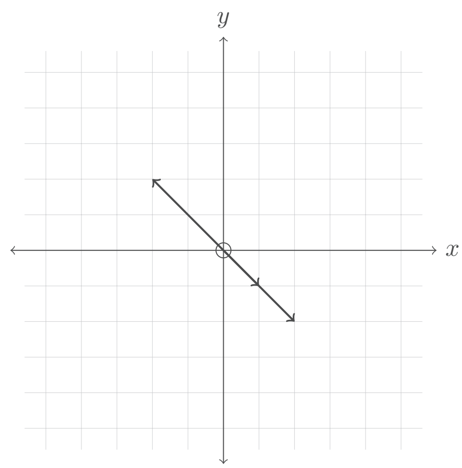
They represent the same line because the vector part of either vector equation is a multiple of the other.

Solution 15:

Line	Vector equation
a	$\lambda < 1, 3 >$ or $\lambda < 2, 6 >$ or...
b	$\lambda < 3, 1 >$ or $\lambda < 6, 2 >$ or...
c	$\lambda < -1, -1 >$ or $\lambda < -2, -2 >$ or...

Solution 16:

	Resultant vector
$1. < 1, -1 >$	$< 1, -1 >$
$2. < 1, -1 >$	$< 2, -2 >$
$-2. < 1, -1 >$	$< -2, 2 >$

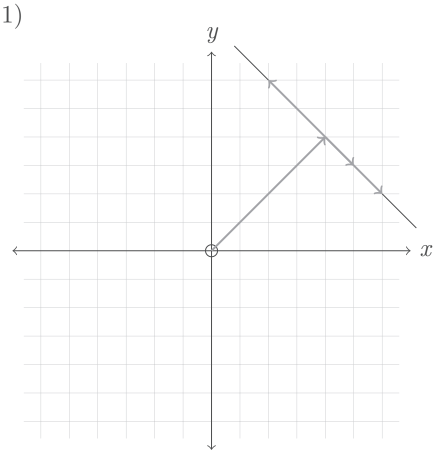


Solution 17:

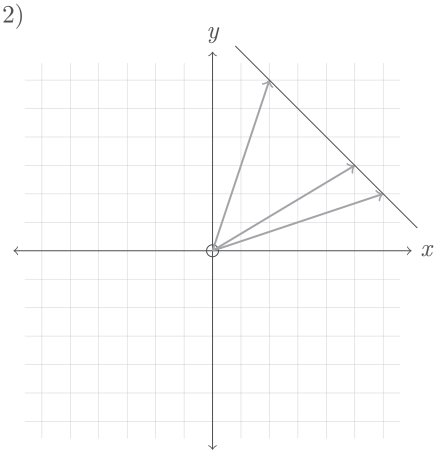
You can use either method to do this:

1. You can draw the two vectors of the vector equation end to end to get each point of the vector equation:
2. You can add the two vectors of the vector equation first and then draw the resultant vector onto the graph

1)



2)



3D Vectors

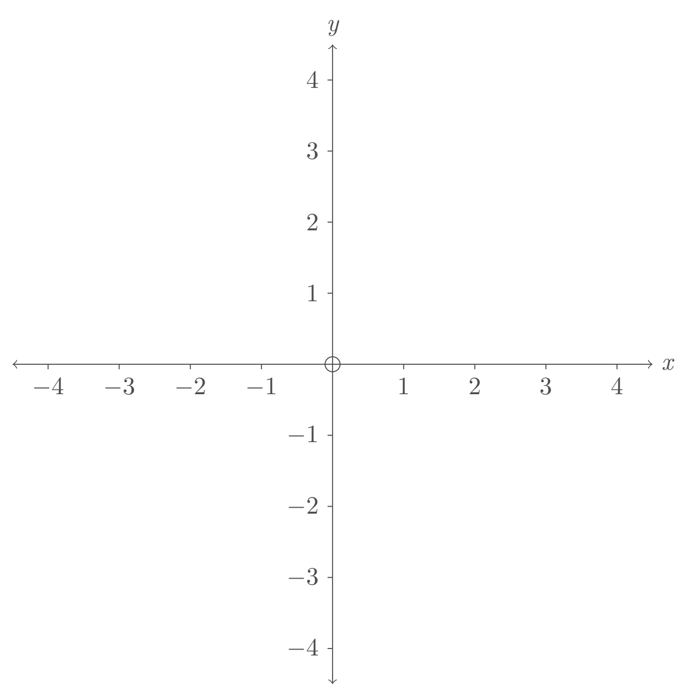
Lesson 2: Welcome to 3D

In the previous module we looked at some of the basic ideas behind vectors. However, all these ideas were presented in 2D. What we are really interested in is 3D and in this module we will look at some of the slight differences between 2D and 3D but will also see that, conceptually, there is very little difference at all.

What does 3D look like?

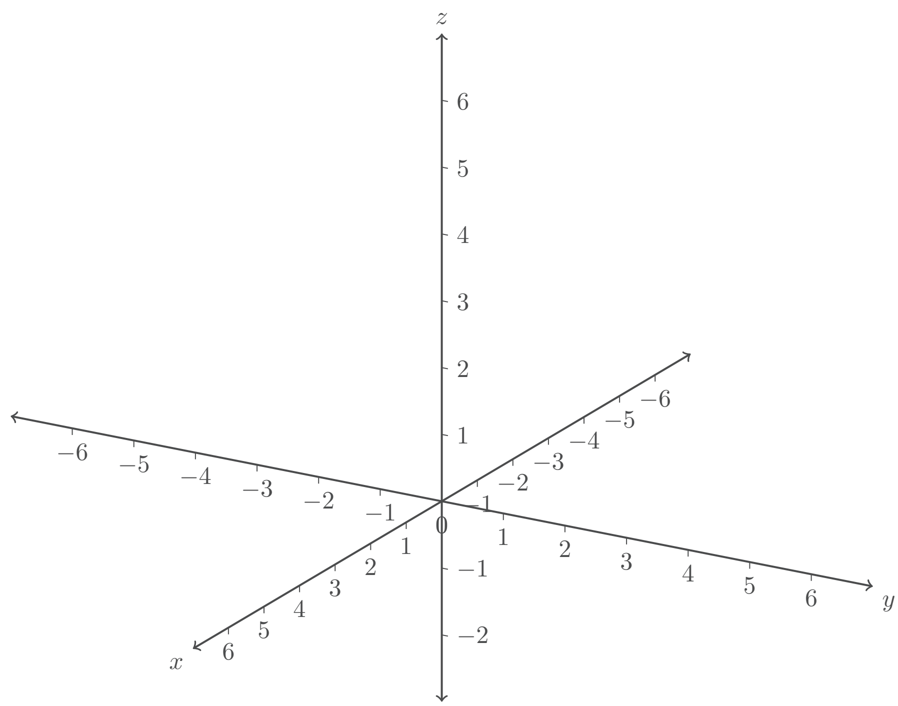
A 2D plane is easy to represent on paper because the surface of paper is 2D!

This is 2D:



To represent 3D we will need to stretch our minds a bit.

This is 3D:



In 3D there is an extra axis labeled z (it's the vertical one, it is the 3rd dimension).

To describe a point in a 2D area you needed 2 coordinates, (x, y). Example:

$$(3, 2)$$

To describe a point in a 3D space you need 3 coordinates, (x, y, z). Example:

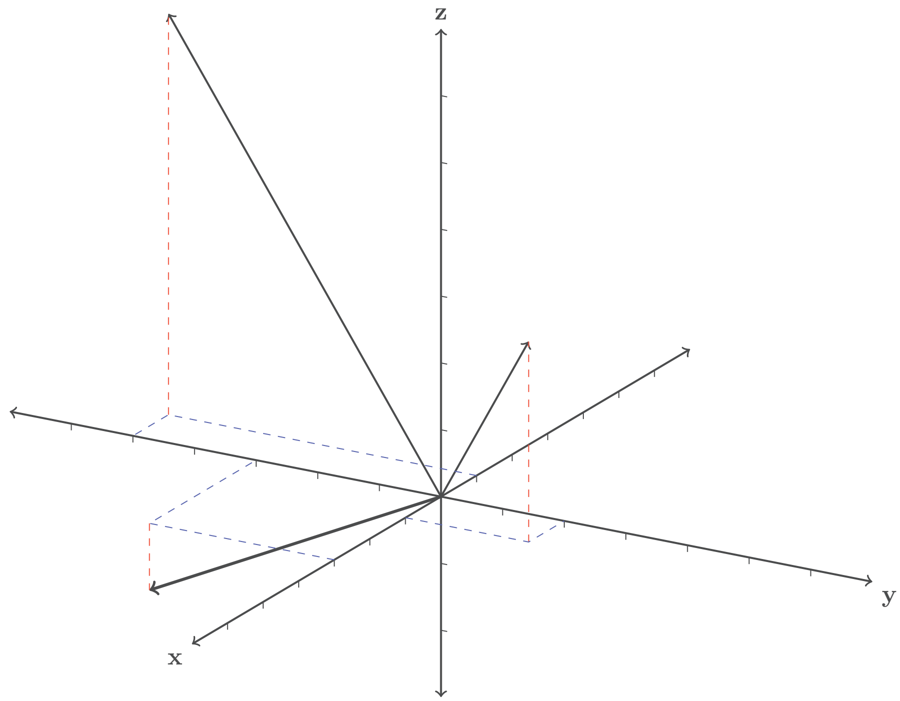
$$(3, 2, 4)$$

So, to describe a vector in a 3D space you need 3 components. Example:

$$< 3, 2, 4 >$$

Here is a picture of a few 3D vectors:

- $< 1, 2, 3 >$
- $< 3, -3, -1 >$
- $< -1, -5, 6 >$



The red and blue dashed lines are included to make it easier to see where the vectors are pointing.

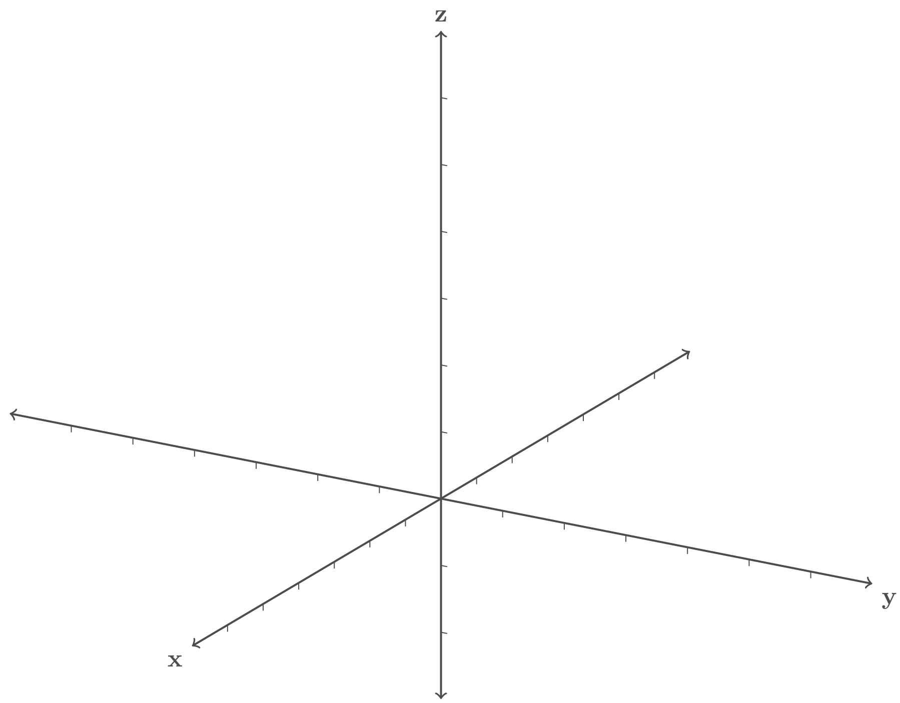
Addition, subtraction and scalar multiplication in 3D

Addition, subtraction and scalar multiplication are all exactly the same as they are in 2D only now we are working with 3 components instead of 2.

Question 1:

Carry out the following operations and sketch them on the set of axes provided: (Your sketches don't have to be perfect as sketching in 3D can be tricky.)

$< 3, 1, 5 > + < 1, 2, -3 >$	
$< 9, 4, 11 > - < 5, 7, 6 >$	
$2\cdot < 1, 3, 1 >$	



Distance in 3D

The distance formula for 3D is just an extension of the theorem of Pythagoras. If you want to find the length of a vector in 2D, say $< 1, 2 >$ for example, this is how you do it:

$$\begin{aligned} | < 1, 2 > | &= \sqrt{(1)^2 + (2)^2} \\ &= \sqrt{5} \end{aligned}$$

Same thing in 3D. If we want to know the length of a 3D vector, say $< 1, 2, 3 >$ for example, this is how we do it:

$$\begin{aligned} | < 1, 2, 3 > | &= \sqrt{(1)^2 + (2)^2 + (3)^2} \\ &= \sqrt{14} \end{aligned}$$

Question 2:
Determine the following:

$\lvert < -3, 9, 1 > \rvert$	
$\lvert < -1, -2, -3 > \rvert$	

Question 3:

Find the distance between the following two vectors: $< 5, 1, -3 >$ and $< 2, 2, 0 >$
(Hint: Subtraction)

Determining a 3D vector line equation

If we are given two points in 3D, we will be able to find the vector equation of the line that runs through them. We only need two points to determine a straight line, even in 3D. But drawing a line through two points is one thing, finding the vector equation that describes that line is another. How do we do it?

There are two things that we need for the equation of any straight line (2D or 3D) in terms of vectors:

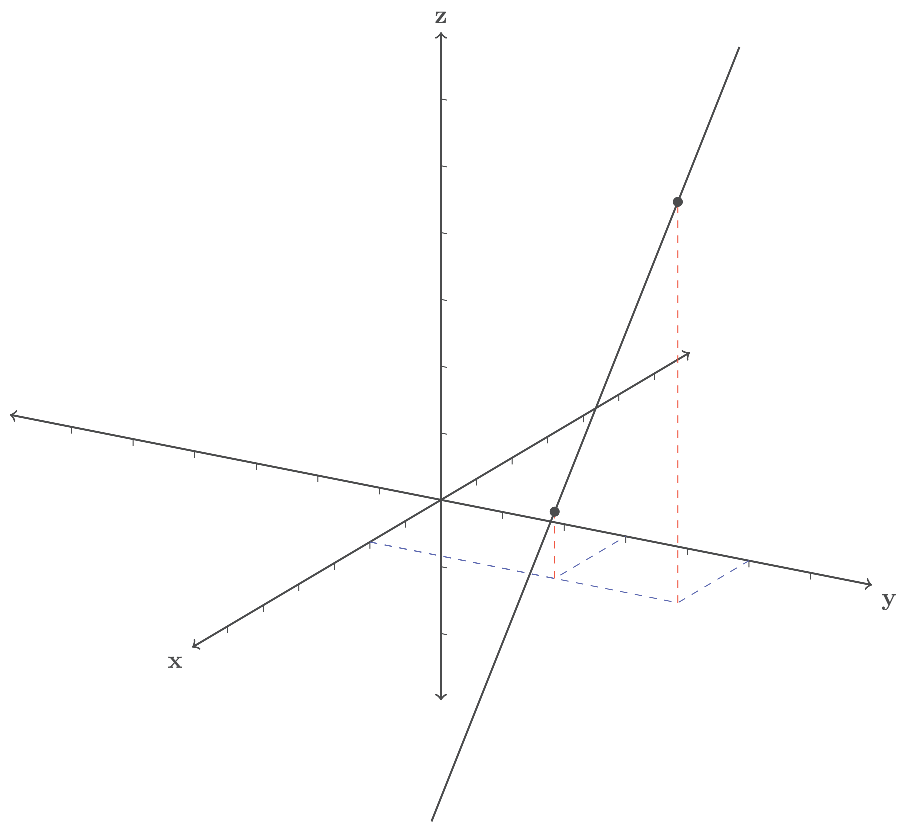
1. First, a vector that goes from the origin to any point on the line. We will call this the position vector.
2. Second, a vector that starts at the end of the first vector and points in the direction of the line. We will call this the direction vector.

If we multiply the direction vector by λ and add it to the position vector, we then get our vector equation for a straight line. This idea works in both 2D and 3D. Now that we know this much, the question becomes, "How do we get the position and direction vectors from the two points that we are given?"

The position vector is easy to get because it will be a vector that goes to either one of the points. Both points are on the line so if the position vector goes to either one of them then it will most certainly be touching the line.

Before we go any further let's start with an example and get a diagram going so that the next part is easier to understand. Let's say that the two given points are $(2, 3, 1)$ and $(2, 5, 6)$. Then the position vector is either $< 2, 3, 1 >$ or $< 2, 5, 6 >$. For our example I'm going to choose $< 2, 5, 6 >$ as my position vector.

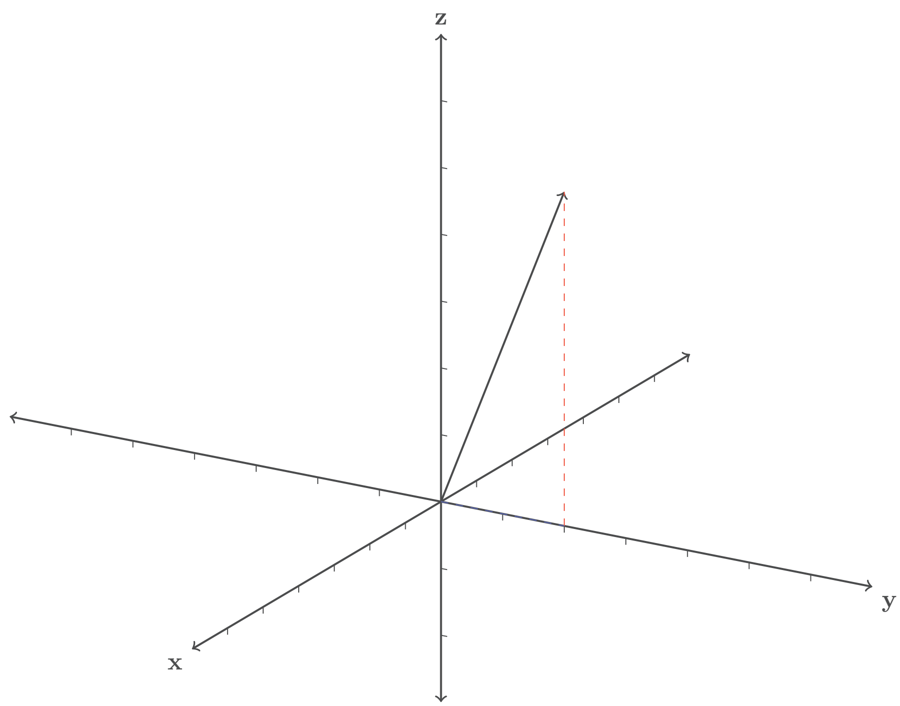
Here is a picture of the line running through the two points:



Our direction vector will have to point along the direction of this line.

To get a vector that points along this direction, we need to think of the two given points as vectors and then subtract them from one another (in any order). Why do we do this? Remember that if you subtract two vectors from one another then the length and direction of the resultant vector is the same as the length and direction of the distance between the ends of the two vectors (we are only interested in the direction though). So if we think of the two points as vectors, we will get the direction vector when we subtract them:

$$< 2, 5, 6 > - < 2, 3, 1 >=< 0, 2, 5 >$$

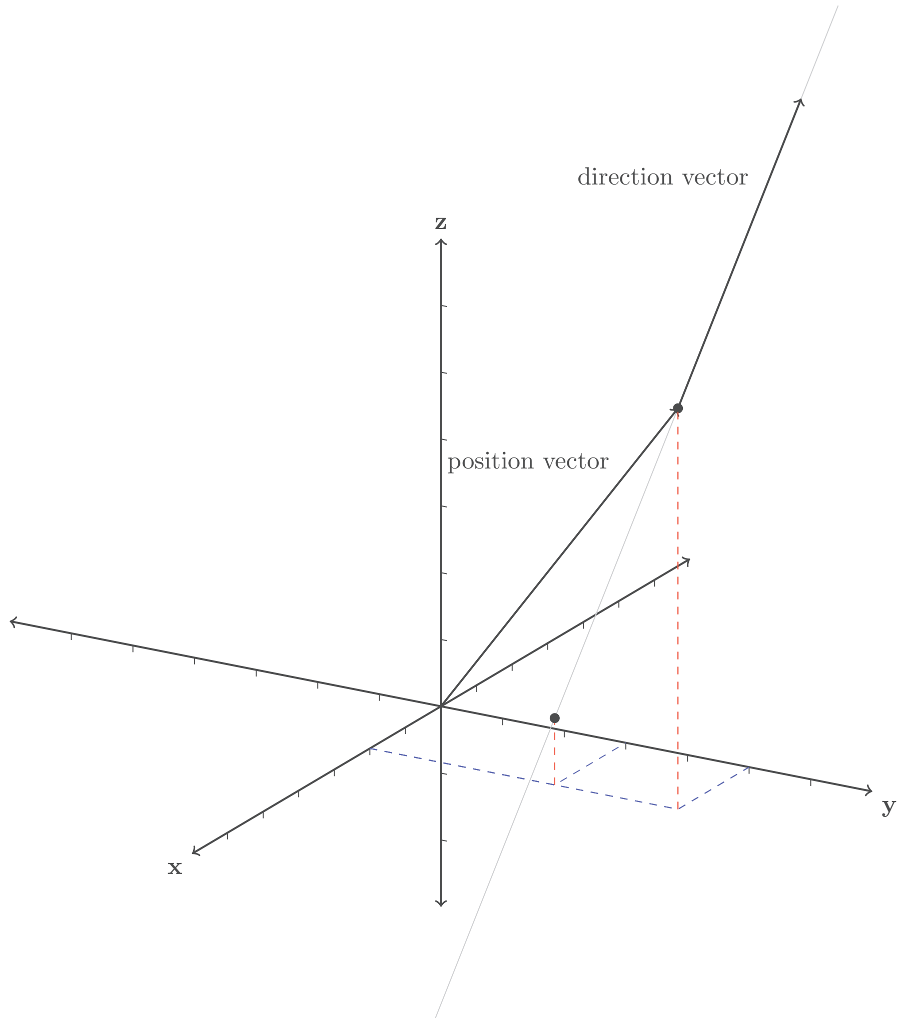


This is the vector $< 0, 2, 5 >$ and it has the same direction as our line running through the two points on the previous diagram. This is our direction vector.

We now have the two vectors that we need to make a vector line equation in the same way as we saw in the previous module.

$$< 2, 5, 6 > +\lambda < 0, 2, 5 >$$

So the final line looks like this:



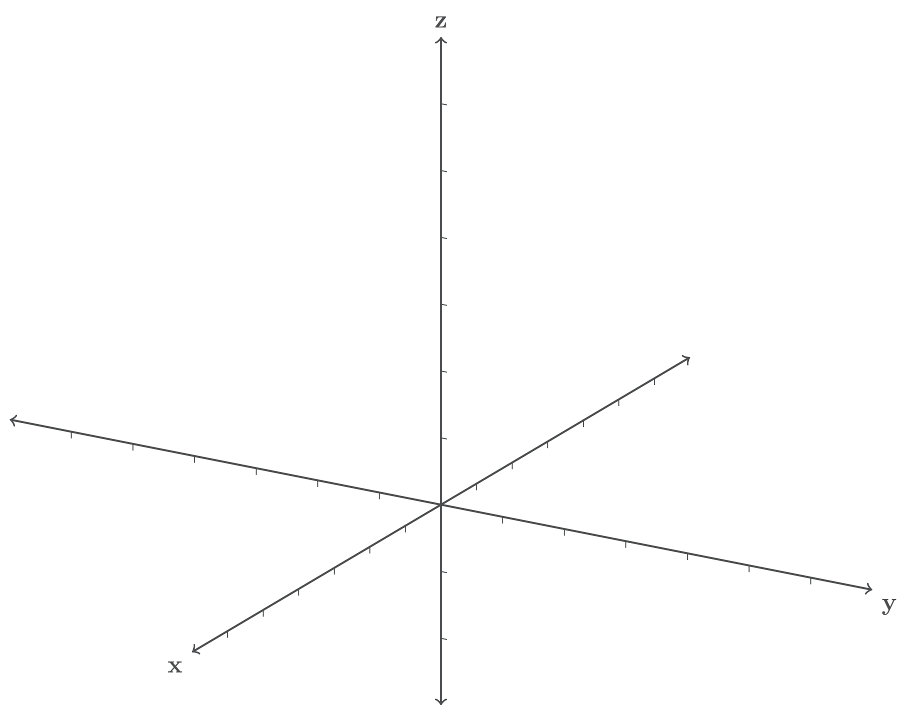
When we write out the vector equation for this line, the direction vector gets a λ infront of it so that it becomes a variable scalar vector.

Question 4:

Find the vector equation of the line that runs through the following two points and, if you want to give it a try, plot the line on the set of axes below:

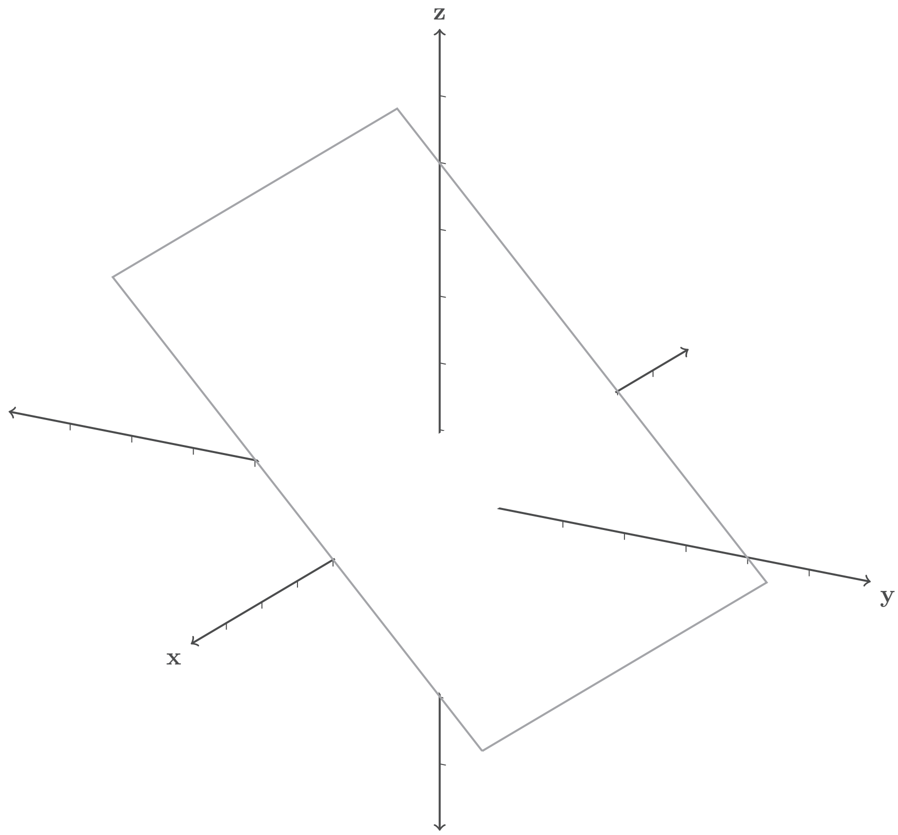
$$(-2, 2, 1) \quad \text{and} \quad (-1, 1, 4)$$

position vector	
direction vector	
3D vector equation	



Looking ahead

Now that you can work in 3D, when you get to university you will see that you don't only have to plot lines. It will be possible to also draw planes! But I can't spoil all the fun now. You will have to wait until next year for planes! Here is a picture of what it looks like:



You will also see that there are two ways to multiply two vectors together, the **dot product** and the **cross product**. If you take the dot product of two vectors you get a scalar and if you take the cross product of two vectors you get another vector! These two concepts are not too difficult and you will get the hang of them quickly.

One last point

In this section we wrote out all our vectors in the following form:

$$< 1, 2, 3 >$$

If we want our vector to be a variable vector then we write all the components as variables.

$$< x, y, z >$$

Note, this is very different to a variable scalar vector, make sure you understand the difference.

A vector can also be written as a variable in one of the following three forms:

$$\mathbf{a} = < x, y, z >$$
$$\bar{a} = < x, y, z >$$
$$\vec{a} = < x, y, z >$$

Solutions

Solution 1:

$< 3, 1, 5 > + < 1, 2, -3 >$	$< 4, 3, 2 >$
$< 9, 4, 11 > - < 5, 7, 6 >$	$< 4, -3, 5 >$
$2\cdot < 1, 3, 1 >$	$< 2, 6, 2 >$

Solution 2:

$\vert < -3, 9, 1 > \vert$	$=\sqrt{(-3)^2 + 9^2 + 1^2}$ $=\sqrt{91}$
$\vert < -1, -2, -3 > \vert$	$=\sqrt{(-1)^2 + (-2)^2 + (-3)^2}$ $=\sqrt{14}$

Solution 3:

Subtract:

$$< 5, 1, -3 > - < 2, 2, 0 >=< 3, -1, -3 >$$

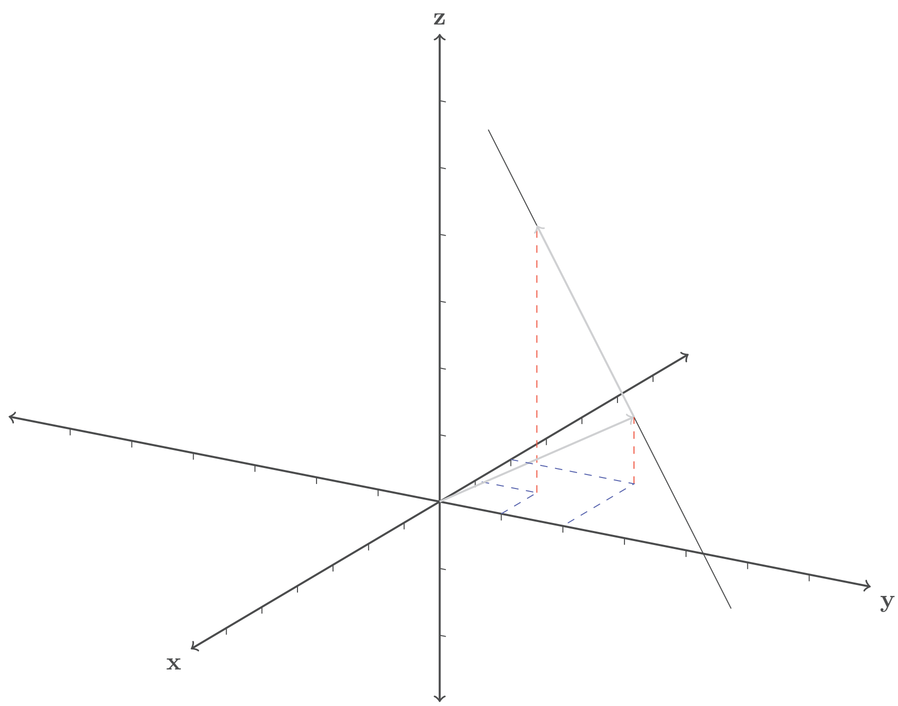
Find the length of this resultant vector:

$$\begin{aligned} | < 3, -1, -3 > | &= \sqrt{3^2 + (-1)^2 + (-3)^2} \\ &= \sqrt{19} \end{aligned}$$

$\sqrt{19}$ is the distance between the two vectors.

Solution 4:

position vector	$< -2, 2, 1 >$ OR $< -1, 1, 4 >$
direction vector	$< -2, 2, 1 > - < -1, 1, 4 >=< -1, 1, -3 >$ OR $< -1, 1, 4 > - < -2, 2, 1 >=< 1, -1, 3 >$
3D vector equation	$< -2, 2, 1 > +\lambda < -1, 1, -3 >$ OR $< -2, 2, 1 > +\lambda < 1, -1, 3 >$ OR $< -1, 1, 4 > +\lambda < -1, 1, -3 >$ OR $< -1, 1, 4 > +\lambda < 1, -1, 3 >$



All four vector equations in the above table map out the same line that is drawn in the graph above. The vector equation that I have used as an example is:

$$< -2, 2, 1 > +\lambda < 1, -1, 3 >$$

The lighter vectors represent the two vectors that make up the vector equation while the dark line represents the line that is mapped out by the vector equation.

Matrices

Lesson 1: Introduction

This lesson is going to start out in a similar way to the first lesson on complex numbers. The reason for this is that I am going to introduce you to a new type of *object* that is similar to a number but isn't one. We can add, subtract, multiply and do other things to it that we can also do with numbers. The new *object* is called a matrix and the plural of matrix is matrices. They come in all different shapes and sizes. Here are some examples:

$$\begin{pmatrix} 1 & 2 & 3 \\ 9 & 8 & 1 \\ 0 & 6 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 9 \end{pmatrix} \quad \begin{pmatrix} -0.5 & 0 \\ 1 & 0.298 \end{pmatrix}$$

Matrices are grids of numbers! That's all there is to it!

You may be thinking,"But why? Why do we suddenly have these grids of numbers called matrices in front of us?" As you progress further with your studies you will find that there are many interesting applications of matrices. They are used in 3D computer graphics, describing tension or strain in material science or engineering and fundamental physics, to name but a few.

Building a matrix

The first and most important step is learning how to build a matrix. A matrix is made up of horizontal rows and vertical columns. We have to choose how many rows and columns we want and which numbers we want in those rows and columns. Here is my choice:

$$\begin{pmatrix} 9 & 0 & 1 \\ -2.5 & 6 & -3 \\ 1.73 & 0 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\begin{array}{cc} & \text{Columns} \\ \text{Rows} & \begin{pmatrix} 9 & 0 & 1 \\ -2.5 & 6 & -3 \\ 1.73 & 0 & 0 \\ 0 & -1 & 2 \end{pmatrix} \end{array}$$

I have chosen 4 rows, 3 columns and in my rows and columns I have put a mix of whole numbers, fractions and zeros. Then just to finish it off you put a big pair of brackets around your rows and columns so that none of the numbers run away. Try building a few of your own matrices.

The numbers in the matrix are called entries; each number is called an entry. Each entry has an address;

- 9 lives at 1st row, 1st column.
- -1 lives at 4th row, 2nd column.

Question 1:

Give the address of the following entries in the above matrix:

Entry	Address
1	
2	
-2.5	

Question 2:

For the same matrix, give each entry living at the following addresses:

Address	Entry
1st row, 2nd column	
3rd row, 1st column	
4th row, 2nd column	

If we want to describe how big the matrix is then we would say that it is a 4x3 matrix.

$$\underset{\text{Number of rows}}{4} \times \underset{\text{Number of Columns}}{3}$$

Question 3:

Look back at the very first five matrices, at the very beginning of the lesson, and determine the size of each matrix is.

Now let's look at how to work with matrices.

Addition

We can't add just any two matrices together; they have to have the same number of rows and columns. In other words they have to have the same size. If they do have the same size then we simply add the corresponding entries. Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

Question 4:

Add the following matrices if possible:

$\begin{pmatrix} -2 & 1 & 0 \\ 3 & -1 & 8 \\ 0 & 1 & 5 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 9 \\ 7 & 1 & -5 \\ 2 & -1 & 1 \end{pmatrix}$	
$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	
$\begin{pmatrix} 1 & 2 \\ 8 & 6 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -3 & -5 \\ 1 & 2 & 9 \end{pmatrix}$	

Question 5:

Why can't we add two matrices if they have different sizes?

Two matrices can also be subtracted from one another if they have the same size. We simply subtract corresponding entries.

Question 6:

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$	

Scalar multiplication

The idea here is exactly the same as for vectors. A scalar is just a number and we are multiplying a matrix by this number. To multiply a matrix by a number we multiply each entry by that number. Example:

$$5 \times \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 9 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 10 \\ 10 & 0 & 5 \\ 0 & 45 & 20 \end{pmatrix}$$

Question 7:

$2 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
$3.5 \times \begin{pmatrix} 1 & -4 & 2 \\ 0 & 0 & 1 \end{pmatrix}$	
$0 \times \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$	

Matrix multiplication

Now we are going to multiply two matrices together. Your first guess as to how this is done is probably to multiply corresponding entries. It would seem like the logical choice since when we add two matrices together we add their corresponding entries, but this guess would be WRONG! Multiplication of two matrices is a little more complex than that. Let's look at an example.

Say I want to multiply the following two matrices together:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 5 & 2 \\ 1 & 0 & 3 \\ 0 & 4 & 1 \end{pmatrix}$$

The technique we use is called 'multiplying sausages' (Note: this is not an official term, I made it up. The official term is dot product, but let's stick with sausage for now.). Think of the rows of the first matrix as horizontal sausages and the columns of the second matrix as vertical sausages.

$$\begin{pmatrix} \boxed{1\ \ 2\ \ 3} \\ \boxed{3\ \ 2\ \ 1} \\ \boxed{0\ \ 1\ \ 0} \end{pmatrix} \times \begin{pmatrix} \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \boxed{\begin{matrix}5\\0\\4\end{matrix}} & \boxed{\begin{matrix}2\\3\\1\end{matrix}} \end{pmatrix}$$

Now we must multiply all the sausages in the 1st matrix with all the sausages in the 2nd matrix. But how do you multiply a sausage with a sausage?

Say we want to multiply the 1st sausage in matrix one with the 1st sausage in matrix two, we do it like this:

$$\boxed{1\ \ 2\ \ 3} \times \boxed{\begin{matrix}0\\1\\0\end{matrix}}$$

$$\begin{aligned} &= (1 \times 0) + (2 \times 1) + (3 \times 0) \\ &= 0 + 2 + 0 \\ &= 2 \end{aligned}$$

So to multiply two sausages we multiply their corresponding terms and then add them up. But what do we do with that 2 we just got? The 2 is an entry in the new matrix that we get when we multiply our two matrices together. But which entry!? Because the 1st sausage was row 1 in matrix one and the 2nd sausage was column 1 in matrix two, our new entry, 2, will be in row 1, column 1 in the new matrix. Example:

$$\begin{pmatrix} 2 & & \\ & & \\ & & \end{pmatrix}$$

All that for just one entry! I know, but it gets easier with practice. Let's go through one more exmple together. Multiply the 1st sausage in matrix one with the 2nd sausage in matrix two:

$$\boxed{1\ \ 2\ \ 3} \times \boxed{\begin{matrix}5\\0\\4\end{matrix}}$$

$$\begin{aligned} &= (1 \times 5) + (2 \times 0) + (3 \times 4) \\ &= 5 + 0 + 12 \\ &= 17 \end{aligned}$$

17 came from multiplying the 1st sausage in matrix one with the 2nd sausage in matrix two therefore it is in row 1, column 2 of our new matrix.

$$\begin{pmatrix} 2 & 17 & \\ & & \\ & & \end{pmatrix}$$

Question 8:

Now, carry on like this until you have multiplied all the sausages in matrix one by all the sausages in matrix two and fill in the table below:

First matrix sausage (row):	Multiplied by second matrix sausage (column):	Calculation:
1	3	
2	1	
2	2	
2	3	
3	1	
3	2	
3	3	

Fill your answers into this matrix:

$$\begin{pmatrix} 2 & 17 & \\ & & \\ \end{pmatrix}$$

Order of multiplication

Now, we can't just multiply any two matrices together. There is a rule. Our next step is to see what this rule is and to do that we start with a question. The first three examples in this question will be straightforward... it's the 4th example that may leave you scratching your head.

Question 9:

Multiply the following matrices if possible:

$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ and $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$	
$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ -4 & 0 & 1 \end{pmatrix}$	
$\begin{pmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ -4 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	

Question 10:

What goes wrong with the 4th exercise in question 9?

____

____

> THE RULE: We can only multiply two matrices together if the horizontal sausages in the 1st matrix have the same number of entries as the vertical sausages in the 2nd matrix.

Another way of saying this is: We can only multiply two matrices together if the number of columns in the 1st matrix is equal to the number of rows in the 2nd matrix.

Look over the examples you have just done to see when this rule is satisfied and when it is not.

Look at the examples again:

- Compare the 1st and 2nd examples.
- Compare the 3rd and 4th examples.

The same two matrices are multiplied together in the 1st and 2nd examples but in different order. The result is different answers.

The same two matrices are multiplied together in the 3rd and 4th examples but in different order. The result is that the matrices in the 3rd example are allowed to be multiplied together but we are not even allowed to multiply the matrices in the fourth example!

The moral of the story: Order is very important in matrix multiplication.

0 and 1

Now that we know a bit about addition and multiplication, we can take a brief look at the matrix equivalents of 0 and 1. What do I mean by matrix equivalents of 0 and 1? Well, we look at how 0 and 1 behave when we add or multiply them with normal numbers and then we try and find matrices that behave in the same way when we add or multiply them with other matrices. This is a strange but simple idea. Read over the following descriptions for clarity:

We know that adding 0 to any number will not change that number. Example:

$$9 + 0 = 9$$

And multiplying any number by 0 will result in 0. Example:

$$9 \times 0 = 0$$

Question 11:

So what kind of matrix behaves like a 0? Try the following exercise and you will see:

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	
$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	
$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	
$\begin{pmatrix} 1 & 0 & 2 \\ 5 & 9 & 6 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	

Any matrix of any size that has 0's for all its entries is a 0 matrix. The following are some examples of 0 matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

And there are many more examples of any size, as long as all the entries are 0.

The same rules about size for addition and multiplication still apply so in the following two examples, even though we are using 0 matrices, they are not the right size for the job:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \text{undefined}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{undefined}$$

What about the matrix equivalent of 1? 1 has the property that if we multiply any number by it, that number stays the same. Example:

$$9 \times 1 = 9$$

So what kind of matrix behaves the same as a 1? You might think that a matrix that has 1's for all its entries behaves like a 1. Let's test this idea:

Question 12:

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	

It turns out that a matrix that has 1's for all its entries DOES NOT behave like a 1. Try the following examples and you will see what kind of matrix does behave like a 1:

Question 13:

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
$\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & 2 & -3 \\ 4 & -5 & 0 \\ 6 & 1 & 2 \end{pmatrix}$	
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 \\ 9 & 8 & 7 & 6 \end{pmatrix}$	
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 9 & 8 & 7 & 6 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	

Question 14:

What do you notice about your answers for each example in the above exercise?

We can now describe what kind of matrix behaves like a 1. If a matrix is square (meaning that it has the same number of rows as columns), has 1's on the diagonal (from top left to bottom right) and 0's everywhere else then it behaves like a 1 when we multiply it by other matrices. We call this kind of matrix an identity matrix (you would think it would be called a 1 matrix... but it's not).

Just remember the size rule:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{is an identity matrix but } \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ is not allowed.}$$

Also, look at the last two examples in the previous question. In both examples the matrix is being multiplied by an identity matrix but when the identity matrix is on the left its size is 2×2 and when it is on the right its size is 4×4. So size does count, even for identity matrices.

Inverse matrices

This is the last idea for this module and we will only skim over it briefly.

The inverse of 9 is $\frac{1}{9}$ and the inverse of $\frac{1}{9}$ is 9.

$$9 \times \frac{1}{9} = 1 \quad \text{and} \quad \frac{1}{9} \times 9 = 1$$

If two numbers are inverses of one another and we multiply them together, in any order, the answer will be 1. The same concept exists with matrices. Some matrices, not all, have an inverse and if we multiply a matrix by its inverse, in any order, then the answer will be an identity matrix (because, as we know, an identity matrix is the matrix equivalent of 1).

Question 15:

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$	
$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	
$\begin{pmatrix} -1 & -1 & 2 \\ 2 & 1 & -2 \\ 1 & 1 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$	
$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & -1 & 2 \\ 2 & 1 & -2 \\ 1 & 1 & -1 \end{pmatrix}$	

In each example the two matrices are inverses of one another! Note; if two matrices are inverses of one another, that does not mean that their entries are inverses of one another.

When you get to university you will learn how to show whether or not a matrix has an inverse and how to find it if it does.

Solutions

Solution 1:

Entry	Address
1	1st row, 3rd column
2	4th row, 3rd column
-2.5	2nd row, 1st column

Solution 2:

Address	Entry
1st row, 2nd column	0
3rd row, 1st column	1.73
4th row, 2nd column	-1

Solution 3:

3×3	2×3	7×5	3×2	2×2

Solution 4:

$\begin{pmatrix} -2 & 1 & 0 \\ 3 & -1 & 8 \\ 0 & 1 & 5 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 9 \\ 7 & 1 & -5 \\ 2 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 9 \\ 10 & 0 & 3 \\ 2 & 0 & 6 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 \\ 8 & 6 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -3 & -5 \\ 1 & 2 & 9 \end{pmatrix}$	Cannot be added together

Solution 5:

To add two matrices you have to add their corresponding entries. If two matrices have different sizes then each matrix will contain entries that do not have corresponding entries in the other.

Solution 6:

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$	$\begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix}$

Solution 7:

$2 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
$3.5 \times \begin{pmatrix} 1 & -4 & 2 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 3.5 & -14 & 7 \\ 0 & 0 & 3.5 \end{pmatrix}$
$0 \times \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Solution 8:

First matrix sausage (row):	Multiplied by second matrix sausage (column):	Calculation:
1	3	$= (1 \times 2) + (2 \times 3) + (3 \times 1)$ $= 2 + 6 + 3$ $= 11$
2	1	$= (3 \times 0) + (2 \times 1) + (1 \times 0)$ $= 0 + 2 + 0$ $= 2$
2	2	$= (3 \times 5) + (2 \times 0) + (1 \times 4)$ $= 15 + 0 + 4$ $= 19$
2	3	$= (3 \times 2) + (2 \times 3) + (1 \times 1)$ $= 6 + 6 + 1$ $= 13$
3	1	$= (0 \times 0) + (1 \times 1) + (0 \times 0)$ $= 0 + 1 + 0$ $= 1$
3	2	$= (0 \times 5) + (1 \times 0) + (0 \times 4)$ $= 0 + 0 + 0$ $= 0$
3	3	$= (0 \times 2) + (1 \times 3) + (0 \times 1)$ $= 0 + 3 + 0$ $= 3$

$$\begin{pmatrix} 2 & 17 & 11 \\ 2 & 19 & 13 \\ 1 & 0 & 3 \end{pmatrix}$$

Solution 9:

$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ and $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 \\ 8 & 3 \end{pmatrix}$
$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ -4 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 2 \\ 0 & -3 & 1 \end{pmatrix}$
$\begin{pmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ -4 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	Cannot be multiplied together

Solution 10:

To be able to multiply two matrices together, the rows of the first matrix must have the same number of entries as the columns of the second matrix so that each entry in the rows of the first matrix has a corresponding entry in the columns of the second matrix.

In the 4th exercise, in question 9, the rows of the first matrix contain 3 entries each while the columns of the second matrix contain 2 entries each. This is no good!

Solution 11:

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 2 \\ 5 & 9 & 6 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Solution 12:

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 3 \\ 7 & 7 \end{pmatrix}$

Solution 13:

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$
$\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & 2 & -3 \\ 4 & -5 & 0 \\ 6 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} -1 & 2 & -3 \\ 4 & -5 & 0 \\ 6 & 1 & 2 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 \\ 9 & 8 & 7 & 6 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 9 & 8 & 7 & 6 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 9 & 8 & 7 & 6 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 9 & 8 & 7 & 6 \end{pmatrix}$

Solution 14:

The matrices that only have 1's on their diagonals and 0's everywhere else don't change the matrices with which they are being multiplied.

Solution 15:

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} -1 & -1 & 2 \\ 2 & 1 & -2 \\ 1 & 1 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & -1 & 2 \\ 2 & 1 & -2 \\ 1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Matrices:

Lesson 2: Putting these matrices to use

As promised, we will now look at a use for matrices. We are going to see how they can be used to solve simultaneous equations. Now, solving simultaneous equations is something you are very familiar with so why should we go through all the trouble of using matrices to do it?

REASON: Solving simultaneous equations of two variables using the method you are used to is easy enough. What about solving simultaneous equations of three or more variables? It is still possible using the method you know but it is more complicated. It becomes even more complicated the more variables your equations have.

If you want to solve simultaneous equations of 2 variables then you need two equations containing those 2 variables.

If you want to solve simultaneous equations of 3 variables then you need three equations containing those 3 variables.

If you want to solve simultaneous equations of 4 variables then you need four equations containing those 4 variables. To get the idea, try solving the following sets of equations:

Question 1:

	x	y	z
$3x + y = 2$ $x + y = 6$			
$x + y + z = 1$ $x + y - z = 0$ $2x - y + 4z = 0$			

You should find the first set of equations easy to solve because you are used to solving equations of two variables. Although you are capable of solving the second set of simultaneous equations don't worry if you can't. In this lesson we will look at a new method which will allow us to solve a set of simultaneous equations of three variables. I will explain this new method using the second set of simultaneous equations from question 1.

$$\begin{aligned} x + y + z &= 1 \\ x + y - z &= 0 \\ 2x - y + 4z &= 0 \end{aligned}$$

Before we get going, I would just like to first get rid of a few unnecessary things from our simultaneous equations:

$$\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & -1 & 4 & 0 \end{array}$$

What has happened!

Compare what I have just written to the simultaneous equations before it. All I have done is written my equations in a less cluttered way. I have decided to only write the coefficients of my simultaneous equations and everything on the left hand side of the = signs is separated from everything on the right of the = signs by a vertical line. We don't have to write the variables, x, y and z because we know that all the numbers in the first column are coefficients of x, all the numbers in the second column are coefficients of y and all the numbers in the third column are coefficients of z. Then we have our vertical line separating the last column from the others because the numbers in the last column are not coefficients, they are the numbers on the right of the = signs.

Nothing has changed! We are only writing our simultaneous equations in a different way. To make it all look pretty let's put some large brackets around the whole lot:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & -1 & 4 & 0 \end{array}\right)$$

What is this? It's a matrix!

Ok, so there is a line between the 3rd and 4th column... just go with it, it won't do any harm. Think of the line as a reminder that our matrix came from a set of simultaneous equations. This kind of matrix is called an **augmented matrix**.

So now we know how to write simultaneous equations as a matrix. This will help us solve them in an easier way but let's first get some practice doing this before we move on.

Question 2:

Rewrite the following sets of simultaneous equations as augmented matrices:

$\begin{aligned} 3x + y &= 2 \\ x + y &= 6 \end{aligned}$	
$\begin{aligned} 9x - 2y + z &= 3 \\ x - z &= 1 \\ x + y + z &= 10 \end{aligned}$	

Now we need to look at how to solve simultaneous equations once we have rewritten them as a matrix. We will carry on using our matrix from before question 2, this one:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & -1 & 4 & 0 \end{array}\right)$$

There is a process that we use to solve our simultaneous equations in this matrix and to go through this process we need three rules. Here they are:

Rule 1: We can swap any two rows around.

$$\text{Example:} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & -1 & 4 & 0 \end{array}\right) \quad \text{can become} \quad \left(\begin{array}{ccc|c} 2 & -1 & 4 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right)$$

I swapped the 1st and 3rd rows around.

Rule 2: We can multiply or divide any row by a constant.

$$\text{Example:} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & -1 & 4 & 0 \end{array}\right) \quad \text{can become} \quad \left(\begin{array}{ccc|c} 5 & 5 & 5 & 5 \\ 1 & 1 & -1 & 0 \\ 2 & -1 & 4 & 0 \end{array}\right)$$

I multiplied the first row by 5.

Rule 3: We can subtract or add any two rows.

$$\text{Example:} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & -1 & 4 & 0 \end{array}\right) \quad \text{can become} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 2 & -1 & 4 & 0 \end{array}\right)$$

I subtracted the 1st row from the 2nd row. Note: the 1st row didn't disappear or change in any way, even though it was subtracted from the second row.

So there are the three rules. But what do we do with them? How do we use our augmented matrix and these three rules to solve our simultaneous equations?

Here is the trick: We have to turn the numbers on the left side of the vertical line into an identity matrix using only our three rules.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & -1 & 4 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}\right) \longleftarrow \text{There will be numbers in this column.}$$

Next we look at HOW to use our three rules to do this. We are going to use quite a few steps so it may seem complicated but each step is actually really simple. As we go along you may wonder how I know to use each rule at each step. There is no set method. The order in which I have chosen to use the rules, in this example, is not the only way. There are many other ways I could have done it. All that matters is that I end up with an identiry matrix. At each step you have to be creative and use the rule that you feel is best for getting one step closer to forming the identity matrix. It comes with practice and practice you will get. For now, follow each step closely and understand why I chose to use each rule when I did.

I start with this:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & -1 & 4 & 0 \end{array}\right)$$

Step 1: Subtract row 1 from row 2:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 2 & -1 & 4 & 0 \end{array}\right)$$

Step 2: Swap rows 2 and 3:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -1 & 4 & 0 \\ 0 & 0 & -2 & -1 \end{array}\right)$$

Step 3: Add row 1 to row 2:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 0 & 5 & 1 \\ 0 & 0 & -2 & -1 \end{array}\right)$$

Step 4: Multiply row 3 by 2,5:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 0 & 5 & 1 \\ 0 & 0 & -5 & -2.5 \end{array}\right)$$

Step 5: Add row 3 to row 2:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & -1.5 \\ 0 & 0 & -5 & -2.5 \end{array}\right)$$

Step 6: Divide row 2 by 3:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -0.5 \\ 0 & 0 & -5 & -2.5 \end{array}\right)$$

Step 7: Divide row 3 by -5:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -0.5 \\ 0 & 0 & 1 & 0.5 \end{array}\right)$$

Step 8: Subtract row 2 from row 1:

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1.5 \\ 1 & 0 & 0 & -0.5 \\ 0 & 0 & 1 & 0.5 \end{array}\right)$$

Step 9: Subtract row 3 from row 1:

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -0.5 \\ 0 & 0 & 1 & 0.5 \end{array}\right)$$

Step 10: Swap rows 1 and 2:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0.5 \end{array}\right)$$

10 steps! That was quite hectic but in the end I got everything on the left of the vertical line to look like an identity matrix. I applied my choice of one of the three rules step by step to achieve this. There is even a name for this. It's called **Gaussian reduction**.

The important thing to understand here is that these steps only worked for this matrix. DO NOT memorize these steps. Each different matrix requires a different set of steps to get the numbers to the left of the vertical line to look like an identity matrix. You have to be creative and decide which steps to use each time. It's an art and you will only get good at it with practice. With time, you will get a feel for which steps to use and when to use them.

Once we have Gauss reduced our matrix, we need to be able to read the solution to our simultaneous equations from the matrix. How do we do that? The solution to your simultaneous equations is in the last column, on the right of the vertical line! The first entry is the solution to x, the second entry is the solution to y and the last entry is the solution to z. To summarise this:

$$\begin{aligned} x &= -0.5 \\ y &= 1 \\ z &= 0.5 \end{aligned}$$

We have just seen HOW to read the solution to our simultaneous equations from a Gauss reduced matrix. Now I will show you WHY the numbers on the right of the vertical line are the solution to the simultaneous equations. I will do this by Gauss reducing another set of simultaneous equations and using them to show you. This Gauss reduction won't be nearly as long as the previous one so don't worry.

This set of simultaneous equations is from question 1:

$$\begin{aligned} 3x + y &= 2 \\ x + y &= 6 \end{aligned}$$

Firstly I turn them into an augmented matrix:

$$\left(\begin{array}{cc|c} 3 & 1 & 2 \\ 1 & 1 & 6 \end{array}\right)$$

Now I use my personal choice of rules to Gauss reduce:

Step 1: I subtract the 2nd row from the 1st row:

$$\left(\begin{array}{cc|c} 2 & 0 & -4 \\ 1 & 1 & 6 \end{array}\right)$$

Step 2: I divide the 1st row by 2:

$$\left(\begin{array}{cc|c} 1 & 0 & -2 \\ 1 & 1 & 6 \end{array}\right)$$

Step 3: I subtract the 1st row from the 2nd row:

$$\left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 8 \end{array}\right)$$

Done! Much easier than the first example.

Question 3:

Look back at your answers to the first example in question 1. What do you notice?

Now I'm going to slowly show you WHY the numbers on the left of the vertical line are the solution to the simultaneous equations.

I'm going to turn $\left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 8 \end{array}\right)$ back into it's equation form by putting the x, y, $+$ and $=$ signs back into their correct places:

$$\begin{aligned} 1x + 0y &= -2 \\ 0x + 1y &= 8 \end{aligned}$$

If I simplify this, I get:

$$\begin{aligned} x &= -2 \\ y &= 8 \end{aligned}$$

And that is exactly the answer you got when you solved these simultaneous equations in the first question. So that's WHY turning the numbers on the left of the vertical line into an identity matrix gives us the solution to our simultaneous equations.

Conclusion: If we turn a set of simultaneous equations into an augmented matrix and Gauss reduce it, then the numbers on the right of the vertical line will be the solution to the simultaneous equations.

Question 4:

Start out by solving some easy two variable simultaneous equations (using matrices, not the old method) and then move on to the more challenging three variable ones. You will have to do your Gauss reduction on a separate sheet of paper and then write your answer in the table provided.

Simultaneous equations	x	y	z
$x + 3y = 5$ $x + 6y = 6$			
$3x + y = 1$ $2x - 5y = 12$			
$2x + 4y = 8$ $4x + 2y = 10$			
$3y + 2z = 7$ $x + 4y - 4z = 3$ $3x + 3y + 8z = 1$			
$5x + 2y + z = 12$ $x + 2z = 7$ $2x + 2y + 2z = 12$			

One last thought

All the matrices that we have seen in this lesson have behaved nicely. However, there are two ways in which things can go wrong.

1: During your Gauss reduction you may find that our matrix suddenly has a row of 0's in it:

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 6 \end{array}\right)$$

If this happens then you will not be able to fully Gauss reduce your matrix. When this happens we call the set of simultaneous equations **indeterminate**. In first year you will learn how to handle indeterminate simultaneous equations.

2: Still, there is something worse that could happen while Gauss reducing your matrix. Something like this:

$$\longrightarrow \left(\begin{array}{ccc|c} 5 & 0 & 1 & 2 \\ 0 & 0 & 0 & 7 \\ 0 & 1 & 3 & 4 \end{array}\right)$$

If at any step in your Gauss Reduction you get a row that only has 0's on the left of the vertical line and some non-zero number on the right of the vertical line then there is no possible solution for the simultaneous equations. Think about it, this is what the middle row is actually saying:

$$0x + 0y + 0z = 7$$

Which simplifies to: $$0 = 7$$

This is nonsense, so this set of simultaneous equations does not have a solution. When this happens we call the set of simultaneous equations **inconsistent**.

You will learn more about indeterminate and inconsistent simultaneous equations next year. The important thing for now is that you know how to Gauss reduce an augmented matrix.

That's all for now but just remember, solving simultaneous equations is only one of the many uses for matrices. They are used for many other things and can be very useful in science, engineering and commerce.

Solutions

Solution 1:

	x	y	z
$\begin{aligned} 3x + y &= 2 \\ x + y &= 6 \end{aligned}$	-2	8	
$\begin{aligned} x + y + z &= 1 \\ x + y - z &= 0 \\ 2x - y + 4z &= 0 \end{aligned}$	-0.5	1	0.5

Solution 2:

$\begin{aligned} 3x + y &= 2 \\ x + y &= 6 \end{aligned}$	$\left(\begin{array}{cc\|c} 3 & 1 & 2 \\ 1 & 1 & 6 \end{array}\right)$
$\begin{aligned} 9x - 2y + z &= 3 \\ x - z &= 1 \\ x + y + z &= 10 \end{aligned}$	$\left(\begin{array}{ccc\|c} 9 & -2 & 1 & 3 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 10 \end{array}\right)$

Solution 3:

The numbers on the right of the vertical line are the same as the results that we got for the first exercise in question 1.

Solution 4:

Simultaneous equations	x	y	z
$\begin{aligned} x + 3y &= 5 \\ x + 6y &= 6 \end{aligned}$	4	$\frac{1}{3}$	
$\begin{aligned} 3x + y &= 1 \\ 2x - 5y &= 12 \end{aligned}$	1	-2	
$\begin{aligned} 2x + 4y &= 8 \\ 4x + 2y &= 10 \end{aligned}$	2	1	
$\begin{aligned} 3y + 2z &= 7 \\ x + 4y - 4z &= 3 \\ 3x + 3y + 8z &= 1 \end{aligned}$	-3	2	$\frac{1}{2}$
$\begin{aligned} 5x + 2y + z &= 12 \\ x + 2z &= 7 \\ 2x + 2y + 2z &= 12 \end{aligned}$	1	2	3

Limits

Lesson 1: Introduction and determining limits graphically

The concept of a limit is one of the most important ideas in first year mathematics as it is used in so many different applications. It forms the foundation of calculus, takes us to the edge of what is possible and even gives us a glimpse into infinity.

So what is a limit?

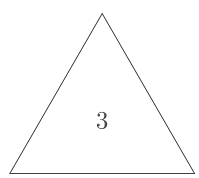
I can't tell you what it is in one definition (I'll leave this for second year). However, I will show you a few ideas and explain the concept of a limit through each one. By the end you should have a good feel for it. Then, for the rest of the lesson, we will practice using limits in different situations.

Getting the idea

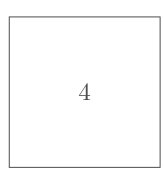
Idea 1

This idea will appeal to those of you who like geometry.

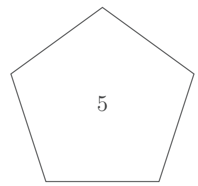
Let me start with a triangle:



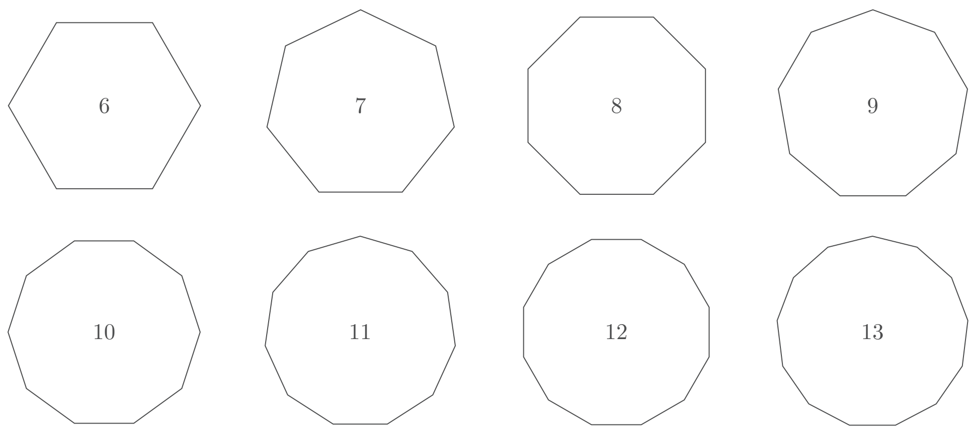
I then add another side to it:



It becomes a square. Adding another side makes it a pentagon:



If I keep adding another side to each shape this is the sequence I get:



What does it look like the shape is becoming?

For every side I add, the shape becomes more and more like a circle. Will it ever become a circle?

How many sides would it take to make it a circle? A thousand? A million? A billion? No matter how many sides we add it can never be a circle because a circle doesn't have any sides!

Couldn't we add an infinite number of sides? No, infinity is not a number. We could keep adding more and more sides but we would never get to infinity. It is impossible to get to infinity so we can only talk about what happens to our shape as the number of sides approaches infinity.

The limit of the shape, as the number of sides approaches infinity, is a circle.

Let's look at another example of a limit.

Idea 2

This idea is for those of you who like sequences.

$$1;\ \frac{1}{2};\ \frac{1}{3};\ \frac{1}{4};\ \frac{1}{5};\ \frac{1}{6};\ \frac{1}{7};\ \frac{1}{8};\ \frac{1}{9};\ ...$$

What is happening to the terms of the sequence? They are getting smaller and smaller the further along we go. But how long do they go on for? Well, forever! The sequence never ends, it goes on to infinity. We can never get to infinity because infinity is not a number. So what does it look like the terms are becoming as we approach infinitely many terms?

In other words, what is the limit of the sequence as we approach infinitely many terms?

The limit of the sequence, as the number of terms approaches infinity, is 0.

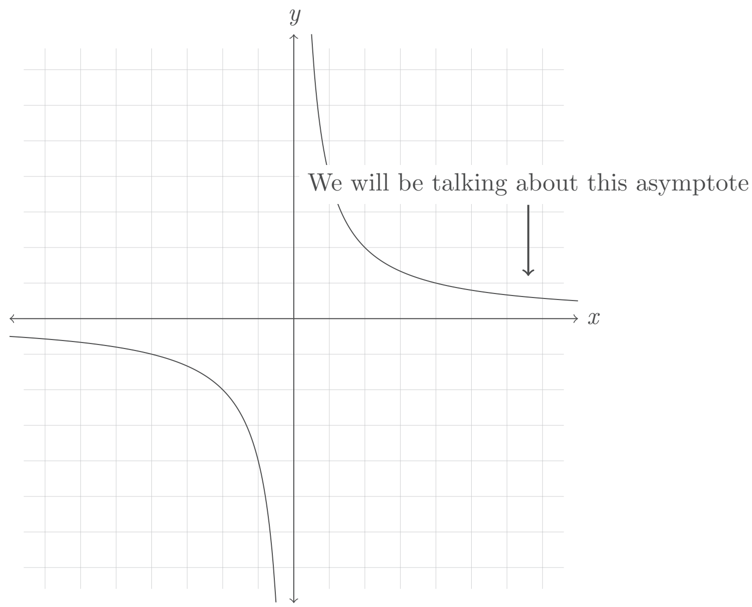
Although the sequence approaches 0, 0 is not a term in the sequence.

Idea 3

This idea is for those of you who like functions.

$$y = \frac{1}{x}$$

This is a standard hyperbola and this is how you sketch it:



We know that a standard hyperbola doesn't ever touch the x or y axes because the x and y axis are its asymptotes. Why though? Let's look at the asymptote that is labelled in the graph.

Think about what happens to the value of y as we make x bigger and bigger. The bigger we make x the smaller y becomes. How big do we have to make x to make y equal to 0?

It doesn't matter how big x gets, y will never equal 0, it only gets closer to 0.

Question 1:

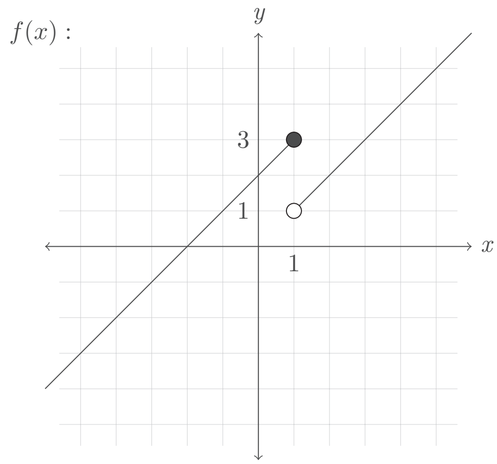
Complete the following statement about the asymptote that is marked out in the graph of the hyperbola above:

The limit of ______ , as ______ approaches ______ , is ______ .

In general, an asymptote is a limit! All this time you have been drawing asymptotes you were also drawing the graphical representation of limits!

Piecewise functions

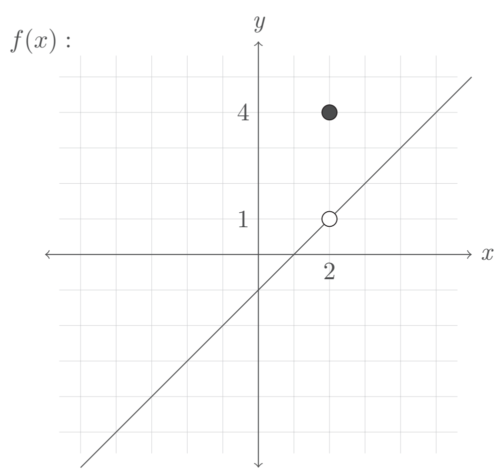
Before we move onto the last idea it would be wise to quickly look at piecewise functions as we will be working with them for the rest of this lesson. Simply put, a piecewise function is a function that is made up of two or more pieces of different types of functions. Example:



This graph contains a bit of $f(x) = x + 2$ and a bit of $f(x) = x$. The $f(x) = x + 2$ bit is defined for values of x up to and including $x = 1$ and the $f(x) = x$ bit is defined for values of x that are greater than 1. This is how we write a function like this:

$$f(x) = \begin{cases} x + 2 & \text{for} \quad x \leq 1 \\ x & \text{for} \quad x > 1 \end{cases}$$

Note how we draw a solid dot at the end of a piece of function if that piece includes the end point and we draw an empty circle if it doesn't. Another type of piecewise function is one that just has one point in a different place. Example:



This graph is described by the function everywhere except at x = 2. At x = 2 the function is equal to 4. We write this function like this:

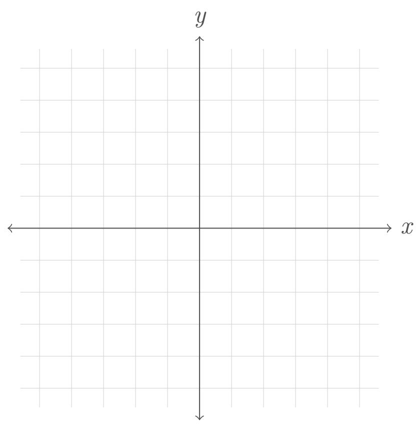
$$f(x) = \begin{cases} x - 1 & \text{for} \quad x \neq 2 \\ 4 & \text{for} \quad x = 2 \end{cases}$$

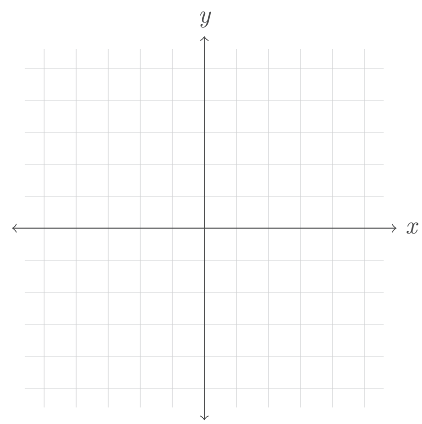
Writing $x \neq 2$ in the function is the same as saying, 'everywhere except 2'.

Question 2:

Plot the following piecewise functions on the axes below:

$$f(x) = \begin{cases} -x & \text{for } x \leq 0 \\ x^2 & \text{for } x > 0 \end{cases} \qquad g(x) = \begin{cases} x & \text{for } x \neq 3 \\ -1 & \text{for } x = 3 \end{cases}$$



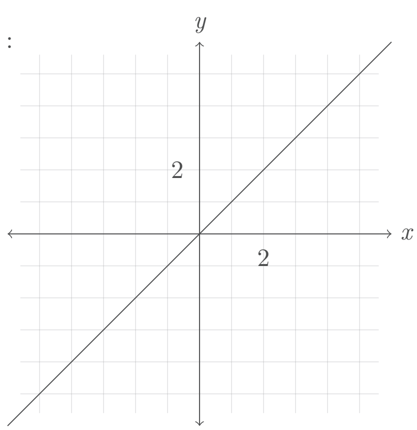


Idea 4

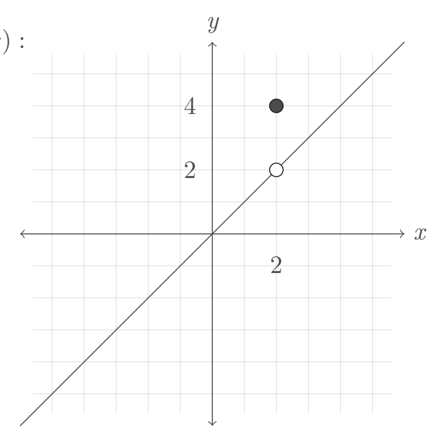
This is the last idea and may seem like the strangest. In all the previous ideas we used limits because we couldn't get an answer to our question. We will now see that we can use limits even when we can get an answer! Even more strange is that sometimes the limit and the answer will be different!

Look at the following two graphs:

$f(x)$:



$g(x)$:



The first graph is a sketch of $f(x) = x$.

The second graph is a sketch of $g(x) = \begin{cases} x & \text{for } x \neq 2 \\ 4 & \text{for } x = 2 \end{cases}$

For the first graph, when $x = 2$, $f(x) = 2$. In other words, $f(2) = 2$. Now I ask you:

What is the limit of $f(x)$ as x approaches 2?

You may be wondering, "Why are we talking about a limit when we already know the answer?" At no point did I say that the limit couldn't exist if the answer did. I only said that the limit could exist if the answer didn't. See the difference? So just because the answer exists doesn't mean we can't find the limit.

Question 3:

As x gets closer to 2, what does $f(x)$ get closer to?

Question 4:

What does $f(x)$ equal at $x = 2$?

Now we will look at the second graph, $g(x)$:

Question 5:

What is the limit of $g(x)$ as x approaches 2?

(Don't think about what $g(x)$ is equal to **at** x=2, think about what it looks like it will be equal to as x gets closer to 2.)

Question 6:

What is $g(x)$ equal to when $x = 2$?

We are now going to practice our idea of a limit by looking at graphs that are similar to the ones in idea 4. But before we do that, I need to introduce some notation.

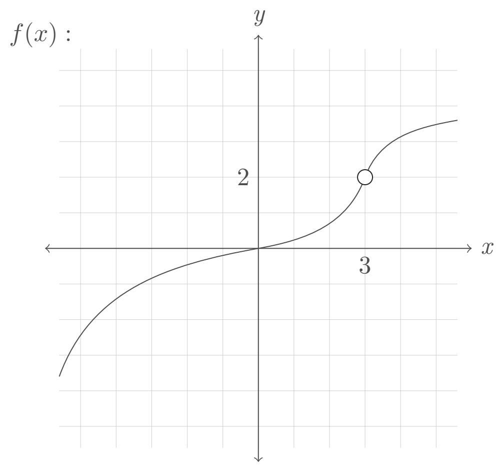
Notation

There must be a shorter way of writing,"The limit of $f(x)$ as x approaches 2". There is! This is it:

$$\lim_{x \to 2} f(x)$$

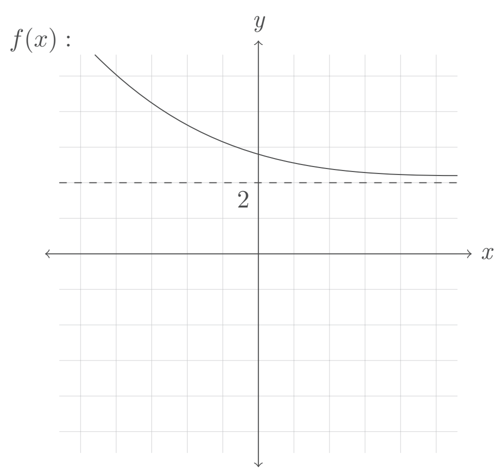
Let's see this notation in action:

Question: Find $\lim_{x \to 3} f(x)$



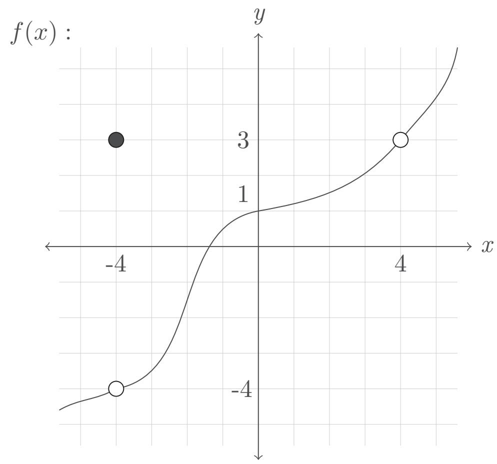
Answer: $\lim_{x \to 3} f(x) = 2$

Question: Find $\lim_{x \to \infty} f(x)$



Answer: $\lim_{x \to \infty} f(x) = 2$

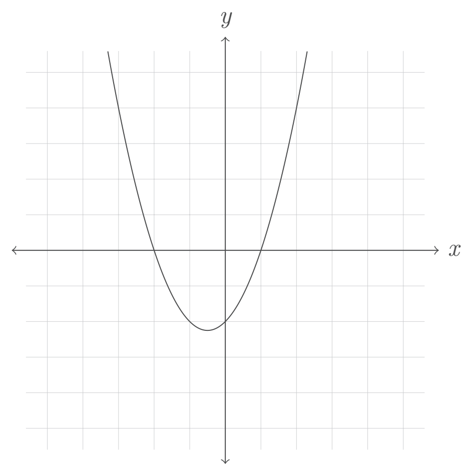
Question 7:



$f(0)$	
$\lim_{x\to 0} f(x)$	
$f(4)$	
$\lim_{x\to 4} f(x)$	
$f(-4)$	
$\lim_{x\to -4} f(x)$	

Question 8:

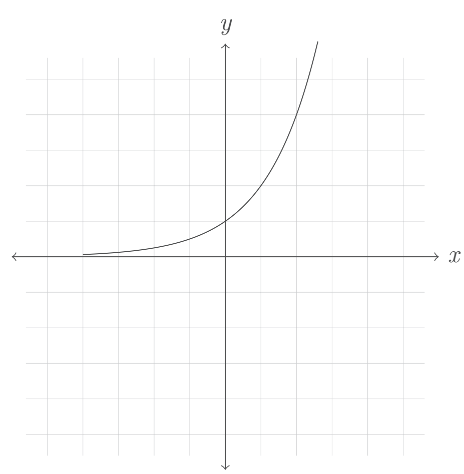
$$f(x) = x^2 + x - 2$$



$\lim_{x\to 2}(x^2 + x - 2)$	

Question 9:

$$f(x) = 2^x$$



$\lim_{x\to -\infty} 2^x$	

Infinity as the answer

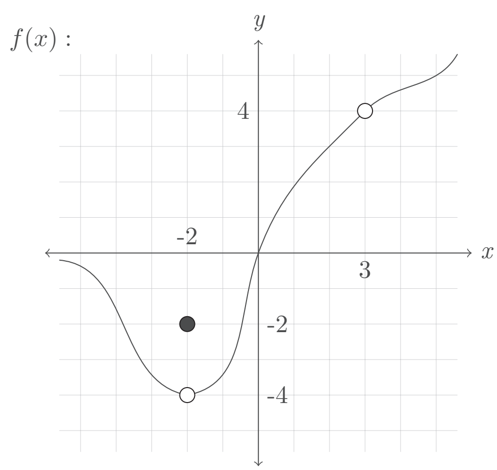
What about $\lim\limits_{x\to\infty} 2^x$? What happens to 2^x as x approaches infinity? 2^x also approaches infinity, so $\lim\limits_{x\to\infty} 2^x = \infty$.

Wait a minute! Infinity is not a number so how can something be equal to it?

When we say that the limit of a function is equal to infinity, as x approaches some value, what we are actually saying is that the limit does not exist... because the function just gets bigger and bigger towards infinity, as x approaches that value. So writing '$\lim\limits_{x\to\infty} 2^x = \infty$' is the same as saying, '2^x just keeps getting bigger and bigger towards infinity, as x approaches ∞, and therefore does not exist'.

Let's carry on with our practice:

Question 10:

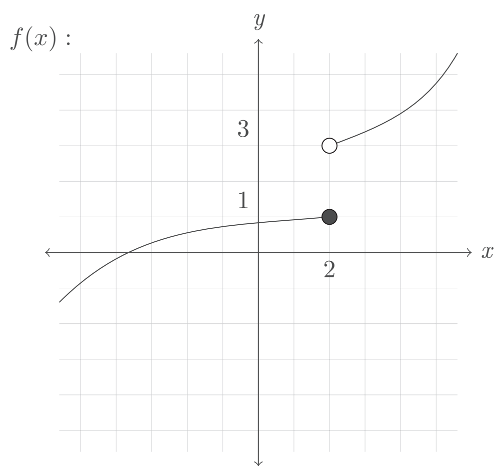


$\lim\limits_{x\to-\infty} f(x)$	
$\lim\limits_{x\to 3} f(x)$	
$\lim\limits_{x\to-2} f(x)$	
$\lim\limits_{x\to\infty} f(x)$	
$f(-2)$	

One-sided limits

Don't answer the next question, just look at it for a little while and think about what is wrong with it before you carry on reading:

Find $\lim_{x \to 2} f(x)$:



Is it 1 or 3? It's neither! If x gets closer to 2 from the left then it looks as though $f(x)$ is getting closer to 1. If x gets closer to 2 from the right then it looks as though $f(x)$ is getting closer to 3. The limit as x approaches 2 does not exist!

In this situation we can only talk about x approaching from one direction. If we want to talk about x approaching 2 from the left, we say:

$$\lim_{x \to 2^-} f(x) \text{ This is a left sided limit}$$

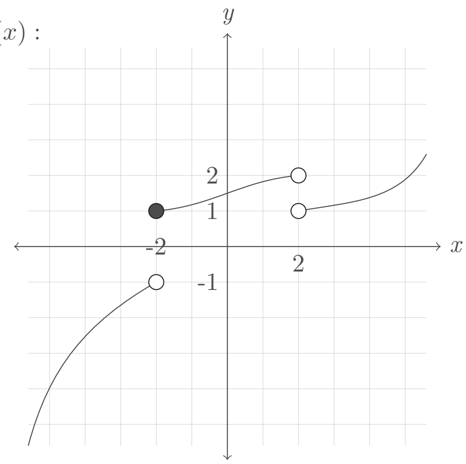
If we want to talk about x approaching 2 from the right we say:

$$\lim_{x \to 2^+} f(x) \text{ This is a right sided limit}$$

These are called one-sided limits. Notice the little $+$ or $-$ at the top right of the 2 in each limit. Try the next practice examples:

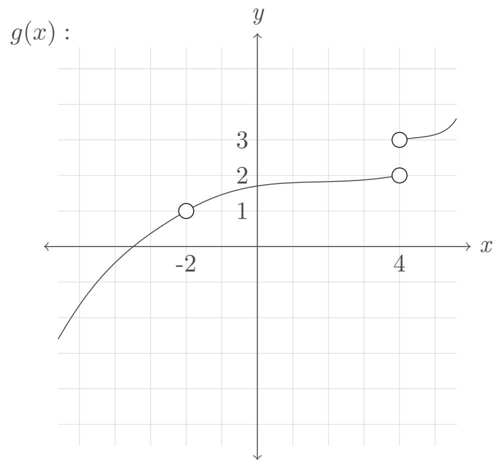
Question 11:

$f(x):$



$\lim\limits_{x\to 2^+} f(x)$	
$\lim\limits_{x\to 2^-} f(x)$	
$\lim\limits_{x\to -2^+} f(x)$	
$\lim\limits_{x\to -2^-} f(x)$	

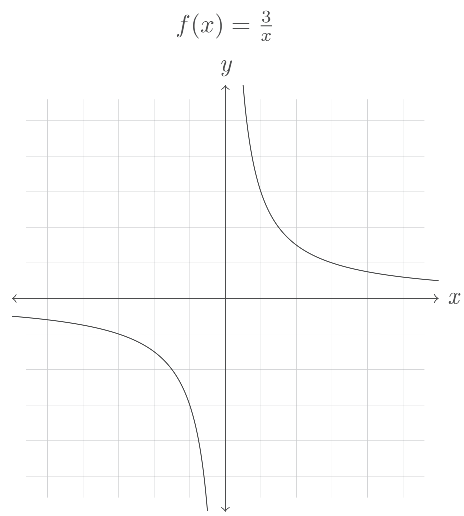
Question 12:



$\lim_{x \to -2^-} g(x)$	
$\lim_{x \to -2^+} g(x)$	
$\lim_{x \to -2} g(x)$	
$\lim_{x \to 4^-} g(x)$	
$\lim_{x \to 4^+} g(x)$	
$\lim_{x \to 4} g(x)$	

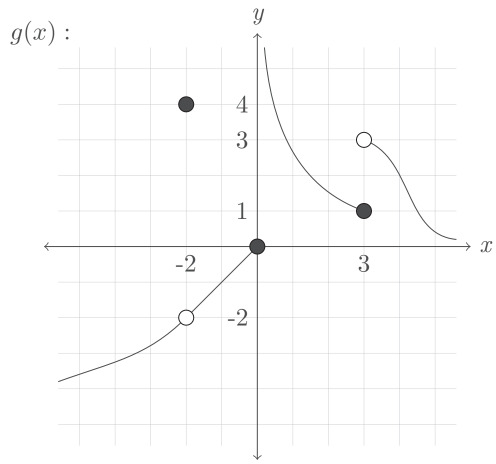
The next two examples combine all the concepts we have learnt about limits so far:

Question 13:



$\lim_{x\to\infty} f(x)$	
$f(0)$	
$\lim_{x\to 0^+} f(x)$	
$\lim_{x\to 0^-} f(x)$	
$\lim_{x\to 0} f(x)$	
$\lim_{x\to -\infty} f(x)$	

Question 14:



$\lim_{x \to -\infty} g(x)$	
$\lim_{x \to -2^-} g(x)$	
$\lim_{x \to -2^+} g(x)$	
$\lim_{x \to -2} g(x)$	
$g(-2)$	
$\lim_{x \to 0^-} g(x)$	
$\lim_{x \to 0^+} g(x)$	
$\lim_{x \to 0} g(x)$	
$\lim_{x \to 3^-} g(x)$	
$\lim_{x \to 3^+} g(x)$	
$\lim_{x \to 3} g(x)$	
$g(3)$	
$\lim_{x \to \infty} g(x)$	

In this lesson we looked at some very basic examples of limits. To find the limits for any of our functions we simply had to look at them and we could immediately tell what the solution was, without having to do any calculation. In the next lesson we will see how to find the limit of a function when we don't have a graph to look at. In these situations we will have to calculate the limit using algebra. The concepts will be exactly the same but we will have to put in more work to get our solutions.

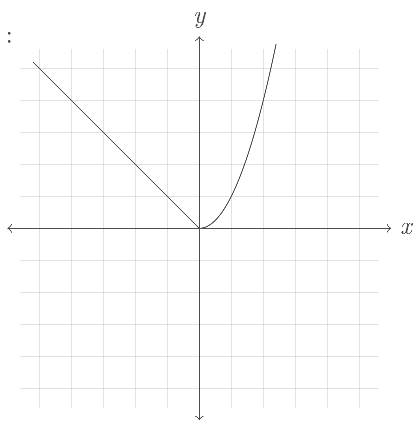
Solutions

Solution 1:

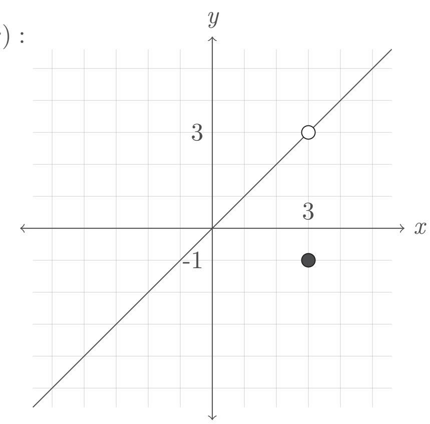
The limit of y **, as** x **approaches** *infinity* **, is** 0 **.**

Solution 2:

$f(x)$:



$g(x)$:



Solution 3:

2

Solution 4:

2

Solution 5:

2

Solution 6:

4

Solution 7:

$f(0)$	1
$\lim\limits_{x \to 0} f(x)$	1
$f(4)$	Does not exist
$\lim\limits_{x \to 4} f(x)$	3
$f(-4)$	3
$\lim\limits_{x \to -4} f(x)$	-4

Solution 8:

$\lim_{x\to 2}(x^2+x-2)$	4

Solution 9:

$\lim_{x\to -\infty} 2^x$	0

Solution 10:

$\lim_{x\to -\infty} f(x)$	0
$\lim_{x\to 3} f(x)$	4
$\lim_{x\to -2} f(x)$	-4
$\lim_{x\to \infty} f(x)$	∞
$f(-2)$	-2

Solution 11:

$\lim_{x\to 2^+} f(x)$	1
$\lim_{x\to 2^-} f(x)$	2
$\lim_{x\to -2^+} f(x)$	1
$\lim_{x\to -2^-} f(x)$	-1

Solution 12:

$\lim_{x \to -2^-} g(x)$	1
$\lim_{x \to -2^+} g(x)$	1
$\lim_{x \to -2} g(x)$	1
$\lim_{x \to 4^-} g(x)$	2
$\lim_{x \to 4^+} g(x)$	3
$\lim_{x \to 4} g(x)$	Does not exist

Solution 13:

$\lim_{x \to \infty} f(x)$	0
$f(0)$	Does not exist
$\lim_{x \to 0^+} f(x)$	∞
$\lim_{x \to 0^-} f(x)$	$-\infty$
$\lim_{x \to 0} f(x)$	Does not exist
$\lim_{x \to -\infty} f(x)$	0

Solution 14:

$\lim_{x\to-\infty} g(x)$	$-\infty$
$\lim_{x\to-2^-} g(x)$	-2
$\lim_{x\to-2^+} g(x)$	-2
$\lim_{x\to-2} g(x)$	-2
$g(-2)$	4
$\lim_{x\to0^-} g(x)$	0
$\lim_{x\to0^+} g(x)$	∞
$\lim_{x\to0} g(x)$	Does not exist
$\lim_{x\to3^-} g(x)$	1
$\lim_{x\to3^+} g(x)$	3
$\lim_{x\to3} g(x)$	Does not exist
$g(3)$	1
$\lim_{x\to\infty} g(x)$	0

Limits

Lesson 2: Determining limits analytically

We will now look at a few helpful techniques and tricks that will allow you to find limits when all we have is a function, without a helpful graph of that function to look at.

Approaching infinity or zero

The first thing that we have to look at when trying to determine a limit just by looking at the function is whether it is approaching 0 or ∞ . You can tell this quite easily without doing much work. Here are some easy examples of what I am talking about:

- $\lim_{x\to\infty} x = \infty$ Obvious!
- $\lim_{x\to 0} x = 0$ Obvious!
- $\lim_{x\to\infty} \frac{1}{x} = 0$ Less obvious.

The bigger x gets in the denominator, the smaller $\frac{1}{x}$ becomes. So as x approaches ∞, $\frac{1}{x}$ approaches 0. If you're not convinced then look at what happens to $\frac{1}{x}$ as I make x bigger:

$$\frac{1}{10}\ldots\ \frac{1}{100}\ldots\ \frac{1}{1000}\ldots\ \frac{1}{1000000}\ldots$$

$\frac{1}{x}$ is getting smaller as x is getting bigger.

- $\lim_{x\to 0^+} \frac{1}{x} = \infty$ Less obvious.

The smaller x gets in the denominator, the bigger $\frac{1}{x}$ becomes. So as x approaches 0 from the right, $\frac{1}{x}$ approaches ∞. Let's see this in action:

$$\frac{1}{0.1} = 10$$
$$\frac{1}{0.01} = 100$$
$$\frac{1}{0.001} = 1000$$
$$\frac{1}{0.000001} = 1000000$$

Before we move on, did you notice that this is a right sided limit? Why are we only looking at the right sided limit in this example? Because, $\lim_{x\to 0} \frac{1}{x}$ does not exist. If we approach from the left we will get a different result to the one that we get when we approach from the right. We know from the previous lesson that this means that $\lim_{x\to 0} \frac{1}{x}$ does not exist. This is what we see if we approach 0 from the left:

$$\begin{aligned} \frac{1}{-0.1} &= -10 \\ \frac{1}{-0.01} &= -100 \\ \frac{1}{-0.001} &= -1000 \\ \frac{1}{-0.000001} &= -1000000 \end{aligned}$$

Clearly we are approaching $-\infty$ when we approach from the left.

If you want to find the limit of a function for a value of x, it is very important to make sure that the left and right sided limits of that function are equal, for that value of x. However, you don't need to worry about left and right sided limits if you are approaching ∞ or $-\infty$ becuase ∞ can only be appraoched from the left and $-\infty$ can only be approached from the right.

- $\lim_{x\to\infty} \dfrac{x+5}{6} = \infty$:

As x approaches ∞, the numerator approaches ∞, while the denominator stays the same. So the whole function must be approaching ∞.

- $\lim_{x\to 3^+} \dfrac{x^2+2}{x-3} = \infty$:

As x approaches 3 from the right, the numerator is getting closer to 11, while the denominator is approaching 0. As the denominator approaches 0, the whole function approaches ∞. If we let x approach 3 from the left, our limit would equal to $-\infty$.

- $\lim_{x\to\infty} \dfrac{1}{x^2+9} = 0$:

The denominator is approaching ∞, so the whole function is approaching 0.

Summary:

- If just the numerator approaches ∞ then the whole function approaches ∞.
- If just the denominator approaches ∞ then the whole function approaches 0.
- If just the numerator approaches 0 then the whole function approaches 0.
- If just the denominator approaches 0 then the whole function approaches ∞.
- If you are looking for the limit of a functon, make sure that the left and right sided limits are equal. Unless, however, you are only looking for the left **or** right sided limit or you are approaching ∞ or $-\infty$.

Question 1:

Find the following limits:

$\lim_{x \to 0} \frac{x}{x+3}$	
$\lim_{x \to \infty} \frac{5}{x}$	
$\lim_{x \to 0^-} \frac{1}{x^2}$	
$\lim_{x \to 0^+} \frac{1}{x^2}$	
$\lim_{x \to 0} \frac{1}{x^2}$	
$\lim_{x \to 0^-} \frac{1}{x^3}$	
$\lim_{x \to 0^+} \frac{1}{x^3}$	
$\lim_{x \to 0} \frac{1}{x^3}$	
$\lim_{x \to 1^+} \frac{x+1}{x-1}$	
$\lim_{x \to 2} \frac{x-2}{x+2}$	
$\lim_{x \to 0} \frac{x^2+x+1}{x}$	
$\lim_{x \to 0^-} \frac{x^2+x+1}{x}$	

But things can go wrong, something funny can happen.

Question 2:

$\lim_{x \to -3} \dfrac{x^2+6x+9}{x+3}$	
$\lim_{x \to \infty} \dfrac{x^2-2x+5}{x^2+x-3}$	

What goes wrong in these two examples? Why can't you find the limits?

If we have a situation in which the numerator and the denominator are both approaching 0 or ∞ then we have something funny going on. There are ways to deal with these funny situations and we will look at them shortly. Let's first look at another easy way to determine a limit and then we will move onto dealing with funny situations.

Direct substitution

If we have a function and x is approaching a value that doesn't cause ANY problems (division by 0, approaching ∞ or something funny happening) then to find the limit of that function as x approaches that value, we need only substitute that value straight into the function. Here is an example:

This is our function:

$$\frac{x^2+5x+6}{x+2}$$

If we want to find the limit of this function as x approaches 4:

$$\lim_{x \to 4} \frac{x^2+5x+6}{x+2}$$

We first ask ourselves, "does substituting 4 into our function cause any problems?"

$$\begin{aligned}\lim_{x \to 4} \frac{x^2+5x+6}{x+2} &= \frac{(4)^2+5(4)+6}{(4)+2} \\ &= \frac{42}{6} \\ &= 7\end{aligned}$$

Since we do not have a 0 or ∞ in either the numerator or the denominator, this substitution does not cause any problems. This method is called **direct substitution**.

Question 3:

Find the limits of the following:

$\lim_{x \to 6} 2^x$	
$\lim_{x \to -1} 3^x$	
$\lim_{x \to 2} (x+1)$	
$\lim_{x \to 1} (x^2 - x - 12)$	
$\lim_{x \to -2} \dfrac{x^2+2x+1}{x+1}$	

We can only use direct substitution if nothing funny happens to our function at the value that x is approaching. Here is an example of something funny happening:

Consider:

$$\lim_{x \to -2} \frac{x^2+5x+6}{x+2}$$

Can we find this limit using direct substitution? Let's try:

$$\frac{(-2)^2+5(-2)+6}{(-2)+2}$$

$$= \frac{0}{0}$$

Definitely something funny happening to our function at $x = -2$. Therefore we can't use direct substitution to solve this limit. So what do we do?

Surgery

When a function has $\frac{0}{0}$ at the value for x that we are approaching then we have to surgically remove the thing that is causing the $\frac{0}{0}$. To perform a surgery we can use any of the following:

- Factorization
- Simplification
- Multiplying the top and bottom by the conjugate of the numerator.

Once you have performed the right surgery on your function then you can substitute the value of x that you are approaching into the function and you will get the limit. If, after performing all possible surgery, you still can't get the limit by substituting the value that x is approaching, then the limit might not exist.

Here are three examples of surgery being performed:

Example 1: Let's start with the example from the previous exercise:

$$\lim_{x \to -2} \frac{x^2 + 5x + 6}{x + 2}$$

We could not solve this using direct substitution because something funny happens at $x = -2$. So we need to perform some surgery.

$$\begin{aligned} &\lim_{x \to -2} \frac{x^2 + 5x + 6}{x + 2} \\ = &\lim_{x \to -2} \frac{(x+3)(x+2)}{x + 2} \\ = &\lim_{x \to -2} (x + 3) \end{aligned}$$

Now that the surgery has been done, we can substitute in the value of x that we are approaching.

$$\begin{aligned} &\lim_{x \to -2} (x + 3) \\ =&((-2) + 3) \\ =&1 \end{aligned}$$

Therefore:

$$\lim_{x \to -2} \frac{x^2 + 5x + 6}{x + 2} = 1$$

That was a simple surgery using only factorization and simplifying. Let's look at a slightly more challenging surgery with some more factorization and simplifying.

Example 2:

$$\begin{aligned} &\lim_{x \to 0} \frac{(3 + x)^2 - 9}{x} && \text{(Something funny happens at 0)} \\ =&\lim_{x \to 0} \frac{9 + 6x + x^2 - 9}{x} \\ =&\lim_{x \to 0} \frac{6x + x^2}{x} \\ =&\lim_{x \to 0} \frac{x(6 + x)}{x} \\ =&\lim_{x \to 0} (6 + x) \\ =&(6 + (0)) \\ =&6 \end{aligned}$$

It's time to see some 'multiplying the top and bottom by the conjugate of the numerator' surgery.

Example 3:

$$
\begin{aligned}
&\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}\\
=&\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}\times\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}\\
=&\lim_{x\to 0}\frac{(x^2+9)-9}{x^2(\sqrt{x^2+9}+3)}\\
=&\lim_{x\to 0}\frac{x^2}{x^2(\sqrt{x^2+9}+3)}\\
=&\lim_{x\to 0}\frac{1}{(\sqrt{x^2+9}+3)}\\
=&\frac{1}{6}
\end{aligned}
$$

Here is one last example to show you just how tricky surgery can get.

Example 4:

$$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)$$

If we substitute $t=0$ into this function we will have two fractions with zeros in their denominators... let the surgery begin!

$$
\begin{aligned}
&\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)\\
=&\lim_{t\to 0}\frac{(t+1)-1}{t(t+1)}\\
=&\lim_{t\to 0}\frac{t}{t(t+1)}\\
=&\lim_{t\to 0}\frac{1}{(t+1)}
\end{aligned}
$$

Surgery complete! Now we can substitute in 0.

$$
\begin{aligned}
&\lim_{t\to 0}\frac{1}{(t+1)}\\
=&\frac{1}{(0)+1}\\
=&1
\end{aligned}
$$

If you have understood the last four surgeries, you should now be a qualified doctor. It's time to operate on your own!

Question 4:

Determine the following limits:

	Result
$\lim_{x\to 1} \dfrac{x^2-1}{x-1}$	
$\lim_{x\to 2} \dfrac{x^2+x-6}{x-2}$	
$\lim_{x\to -2} \dfrac{x^2-x-6}{x+2}$	
$\lim_{x\to 7} \dfrac{\sqrt{x+2}-3}{x-7}$	

$\lim_{t \to -3} \frac{t^2 - 9}{2t^2 + 7t + 3}$	
$\lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$	
$\lim_{x \to -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x}$	
$\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h}$	

Surgery for infinity over infinity

I have one more type of surgery for you. This type of surgery is used when we have a rational function, like this one:

$$\frac{x^2 + x - 3}{2x^2 - x + 5}$$

and you want to find the limit as x approaches infinity:

$$\lim_{x \to \infty} \frac{x^2 + x - 3}{2x^2 - x + 5}$$

Something funny happens here because both the numerator and the denominator are approaching infinity. If you are thinking that the infinity on the top cancels with the infinity on the bottom and your answer is 1 then you are wrong. Infinity is not a number so you can't cancel it with infinity. Definitely time for some surgery... but what kind?

Divide the top and bottom of the rational function by the highest power of x in the denominator. In this case that would be x^2 . So we have:

$$\begin{aligned}
&\lim_{x\to\infty} \frac{\left(\frac{x^2+x-3}{x^2}\right)}{\left(\frac{2x^2-x+5}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{\left(\frac{x^2}{x^2}+\frac{x}{x^2}-\frac{3}{x^2}\right)}{\left(\frac{2x^2}{x^2}-\frac{x}{x^2}+\frac{5}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{\left(1+\frac{1}{x}-\frac{3}{x^2}\right)}{\left(2-\frac{1}{x}+\frac{5}{x^2}\right)}
\end{aligned}$$

Now what happens to the top and bottom as x approaches infinity?

If we look at the numerator:

- $-\frac{3}{x^2}$ gets closer to 0.
- $\frac{1}{x}$ gets closer to 0.
- 1 just stays the same.

If we look at the denominator:

- $\frac{5}{x^2}$ gets closer to 0.
- $-\frac{1}{x}$ gets closer to 0.
- 2 just stays the same.

Therefore:

$$\lim_{x\to\infty} \frac{x^2+x-3}{2x^2-x+5} = \frac{1}{2}$$

Question 5:

Determine the following limits:

	Result
$\lim_{x\to\infty} \frac{x^3+5x}{2x^3-x^2+4}$	
$\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$	
$\lim_{t\to-\infty} \frac{t^2+2}{t^3+t^2-1}$	

One last thought

If we have the following kind of limit:

$$\lim_{x \to 0} 3$$

then the answer is 3. This may seem strange but, if you think about it, why does 3 care about what x is doing? It doesn't matter what x does, 3 is 3.

Example:

$$\lim_{x \to 1} 3 = 3$$

Or

$$\lim_{x \to -5} 3 = 3$$

This may seem like a silly idea but it is useful to know.

Question 6:

Determine the following limits:

$\lim_{x \to 7} 10$	
$\lim_{x \to -\infty} 6$	
$\lim_{x \to 0} \left(-\frac{x}{x}\right)$	

What I have shown you in this lesson is just a taste of what's to come. There are many more methods and ideas that are used to solve limits and the stronger your understanding of limits becomes, the more naturally you will be able to find them. You will be exposed to limits in a more mathematical way when you get to university but if you have understood the basic ideas presented in this lesson then you are well on your way to understanding whatever university has to throw at you.

Solutions

Solution 1:

$\lim_{x \to 0} \frac{x}{x+3}$	0
$\lim_{x \to \infty} \frac{5}{x}$	0
$\lim_{x \to 0^-} \frac{1}{x^2}$	∞
$\lim_{x \to 0^+} \frac{1}{x^2}$	∞
$\lim_{x \to 0} \frac{1}{x^2}$	∞
$\lim_{x \to 0^-} \frac{1}{x^3}$	$-\infty$
$\lim_{x \to 0^+} \frac{1}{x^3}$	∞
$\lim_{x \to 0} \frac{1}{x^3}$	Undefined
$\lim_{x \to 1^+} \frac{x+1}{x-1}$	∞
$\lim_{x \to 2} \frac{x-2}{x+2}$	0
$\lim_{x \to 0} \frac{x^2+x+1}{x}$	Undefined
$\lim_{x \to 0^-} \frac{x^2+x+1}{x}$	$-\infty$

Solution 2:

$\lim_{x \to -3} \frac{x^2+6x+9}{x+3}$	The top AND bottom of this fraction get closer to 0, as x approaches 3, so we don't know if the fraction is getting smaller or bigger, as x approaches 3.
$\lim_{x \to \infty} \frac{x^2-2x+5}{x^2+x-3}$	The top AND bottom of this fraction get closer to infinity, as x approaches infinity, so we don't know if the fraction is getting smaller or bigger, as x approaches infinity.

Solution 3:

$\lim_{x\to 6} 2^x$	64
$\lim_{x\to -1} 3^x$	$\frac{1}{3}$
$\lim_{x\to 2}(x+1)$	3
$\lim_{x\to 1}(x^2-x-12)$	-12
$\lim_{x\to -2} \dfrac{x^2+2x+1}{x+1}$	-1

Solution 4:

	Result
$\lim_{x\to 1} \dfrac{x^2-1}{x-1}$	$=\lim_{x\to 1} \dfrac{(x-1)(x+1)}{x-1}$ $=\lim_{x\to 1}(x+1)$ $=2$
$\lim_{x\to 2} \dfrac{x^2+x-6}{x-2}$	$=\lim_{x\to 2} \dfrac{(x+3)(x-2)}{x-2}$ $=\lim_{x\to 2}(x+3)$ $=5$
$\lim_{x\to -2} \dfrac{x^2-x-6}{x+2}$	$=\lim_{x\to -2} \dfrac{(x-3)(x+2)}{x+2}$ $=\lim_{x\to -2}(x-3)$ $=-5$
$\lim_{x\to 7} \dfrac{\sqrt{x+2}-3}{x-7}$	$=\lim_{x\to 7} \dfrac{\sqrt{x+2}-3}{x-7} \times \dfrac{\sqrt{x+2}+3}{\sqrt{x+2}+3}$ $=\lim_{x\to 7} \dfrac{(x+2)-9}{(x-7)(\sqrt{x+2}+3)}$ $=\lim_{x\to 7} \dfrac{x-7}{(x-7)(\sqrt{x+2}+3)}$ $=\lim_{x\to 7} \dfrac{1}{(\sqrt{x+2}+3)}$ $=\dfrac{1}{6}$

$\lim\limits_{t\to -3} \dfrac{t^2-9}{2t^2+7t+3}$	$\begin{aligned} &= \lim_{t\to -3} \frac{(t-3)(t+3)}{(2t+1)(t+3)} \\ &= \lim_{t\to -3} \frac{t-3}{2t+1} \\ &= \frac{6}{5} \end{aligned}$
$\lim\limits_{h\to 0} \dfrac{(2+h)^2-4}{h}$	$\begin{aligned} &= \lim_{h\to 0} \frac{4+4h+h^2-4}{h} \\ &= \lim_{h\to 0} \frac{4h+h^2}{h} \\ &= \lim_{h\to 0} \frac{h(4+h)}{h} \\ &= \lim_{h\to 0} (4+h) \\ &= 4 \end{aligned}$
$\lim\limits_{x\to -4} \dfrac{\frac{1}{4}+\frac{1}{x}}{4+x}$	$\begin{aligned} &= \lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} \times \frac{4x}{4x} \\ &= \lim_{x\to -4} \frac{\frac{4x}{4}+\frac{4x}{x}}{(4+x)4x} \\ &= \lim_{x\to -4} \frac{x+4}{(4+x)4x} \\ &= \lim_{x\to -4} \frac{1}{4x} \\ &= -\frac{1}{16} \end{aligned}$
$\lim\limits_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}$	$\begin{aligned} &= \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} \times \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1} \\ &= \lim_{h\to 0} \frac{(1+h)-1}{h(\sqrt{1+h}+1)} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{1+h}+1)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1} \\ &= \frac{1}{2} \end{aligned}$

Solution 5:

	Result
$\lim\limits_{x\to\infty} \dfrac{x^3+5x}{2x^3-x^2+4}$	$= \lim\limits_{x\to\infty} \dfrac{\frac{x^3}{x^3}+\frac{5x}{x^3}}{\frac{2x^3}{x^3}-\frac{x^2}{x^3}+\frac{4}{x^3}}$ $= \lim\limits_{x\to\infty} \dfrac{1+\frac{5}{x^2}}{2-\frac{1}{x}+\frac{4}{x^3}}$ $=\dfrac{1}{2}$
$\lim\limits_{x\to\infty} \dfrac{3x^2-x-2}{5x^2+4x+1}$	$= \lim\limits_{x\to\infty} \dfrac{\frac{3x^2}{x^2}-\frac{x}{x^2}-\frac{2}{x^2}}{\frac{5x^2}{x^2}+\frac{4x}{x^2}+\frac{1}{x^2}}$ $= \lim\limits_{x\to\infty} \dfrac{3-\frac{1}{x}-\frac{2}{x^2}}{5+\frac{4}{x}+\frac{1}{x^2}}$ $=\dfrac{3}{5}$
$\lim\limits_{t\to-\infty} \dfrac{t^2+2}{t^3+t^2-1}$	$= \lim\limits_{t\to-\infty} \dfrac{\frac{t^2}{t^3}+\frac{2}{t^3}}{\frac{t^3}{t^3}+\frac{t^2}{t^3}-\frac{1}{t^3}}$ $= \lim\limits_{t\to-\infty} \dfrac{\frac{1}{t}+\frac{2}{t^3}}{1+\frac{1}{t}-\frac{1}{t^3}}$ $=0$

Solution 6:

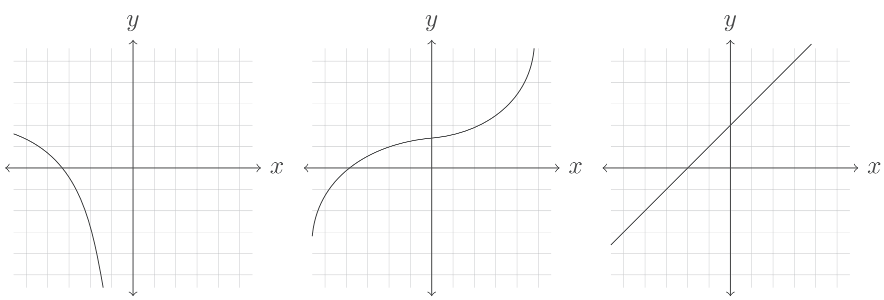
$\lim\limits_{x\to7} 10$	10
$\lim\limits_{x\to-\infty} 6$	6
$\lim\limits_{x\to0} \left(-\frac{x}{x}\right)$	-1 (the x's cancel out)

Continuity

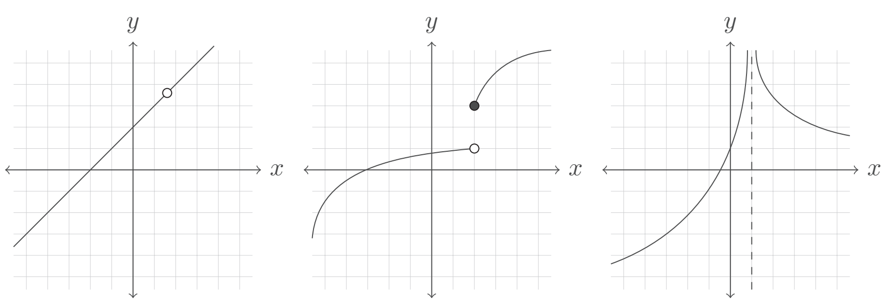
Lesson 1: Introduction

This is the basic idea: If you can draw a graph without lifting your pencil then it is continuous. If you have to lift your pencil to draw the whole graph then it is discontinuous.

These graphs are continuous:



These graphs are discontinuous:



Hole Gap Vertical Asymptote

Each of these discontinuous graphs is an example of the 3 types of discontinuities.

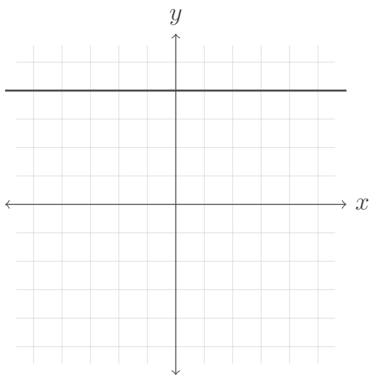
Hole: This discontinuity happens when a single point is missing from the function. Its proper name is **removable discontinuity**. You will see why later.

Gap: Also called a **jump discontinuity** because there is a space separating the two parts of the graph that you have to jump across.

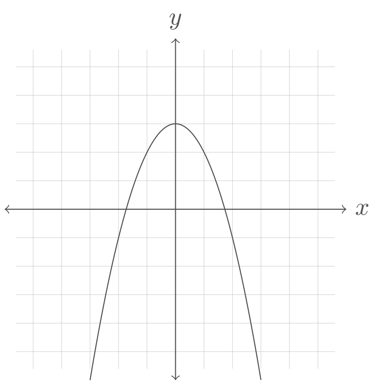
Vertical Asymptote: This type of discontinuity happens when two parts of the graph are separated by a vertical asymptote.

Question 1:

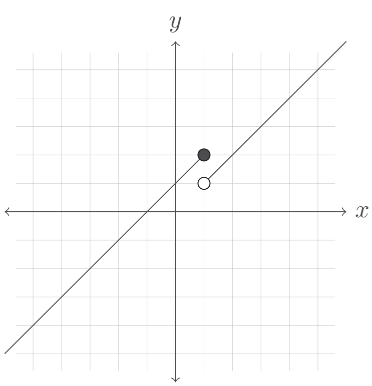
Determine whether the following graphs are continuous or not. If a graph is discontinuous, say what type of discontinuity it has:



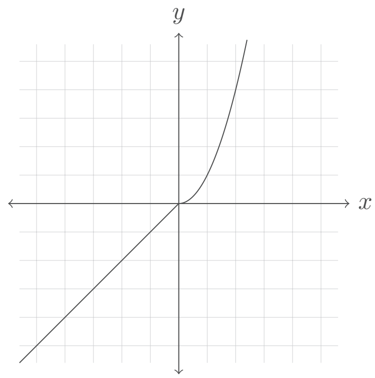
$y = 4$



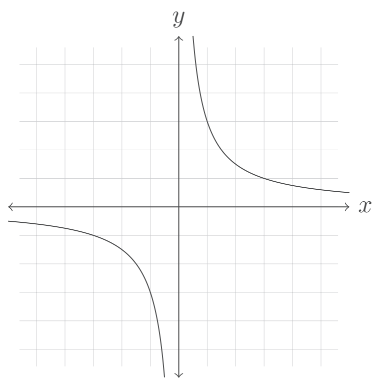
$f(x) = -x^2 + 3$



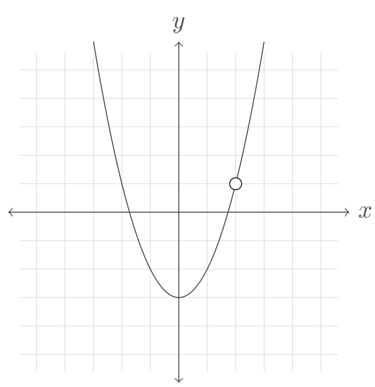
$$f(x) = \begin{cases} x & \text{for} \quad x > 1 \\ x+1 & \text{for} \quad x \le 1 \end{cases}$$



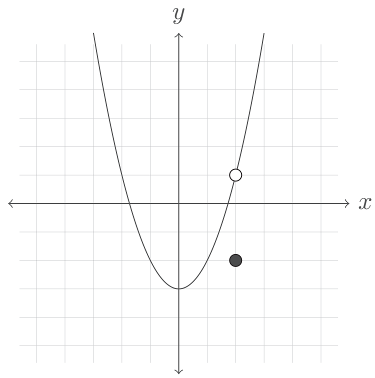
$$f(x) = \begin{cases} x & \text{for} \quad x < 0 \\ x^2 & \text{for} \quad x \geq 0 \end{cases}$$



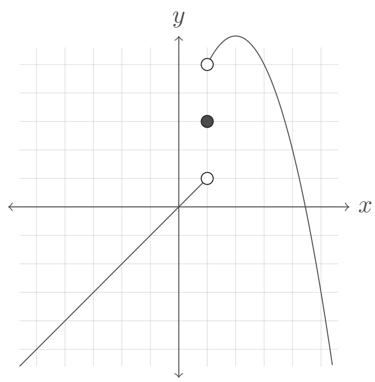
$$f(x) = \frac{3}{x}$$



$$f(x) = x^2 - 3 \quad \text{for} \quad x \neq 2$$



$$f(x) = \begin{cases} x^2 - 3 & \text{for} \quad x \neq 2 \\ -2 & \text{for} \quad x = 2 \end{cases}$$



$$f(x) = \begin{cases} x & \text{for} \quad x < 1 \\ 2 & \text{for} \quad x = 1 \\ -x^2 + 4x + 2 & \text{for} \quad x > 1 \end{cases}$$

So, to tell if a function is discontinuous, all you have to do is look at its graph and see if there are any holes, gaps or vertical asymptotes.

Now what happens if I give you a function but not its graph to look at? Then I ask you if it is continuous or discontinuous. We will need a way of working this out.

Here's how the thinking goes:

Either our function is well behaved or it is doing something funny.

These are examples of well behaved functions:

$$f(x) = x + 1 \qquad f(x) = -x^2 + 2x + 1 \qquad f(x) = 5^x + 8 \qquad f(x) = 3$$

These functions are well behaved because no matter what value you choose x to be, nothing funny happens. If your function is well behaved then you know it is continuous.

Funny functions have division by zero, are not defined for certain values of x or consist of a mix of two or more functions on different intervals (piecewise functions).

Question 2:

For each function briefly describe why something funny is happening

- $f(x) = \dfrac{x+1}{x}$

- $f(x) = \dfrac{5}{x}$

- $f(x) = \dfrac{x^2 - 5x + 6}{x - 3}$

- $f(x) = \begin{cases} x & \text{for} \quad x < 0 \\ x + 1 & \text{for} \quad x \geq 0 \end{cases}$

- $f(x) = x^2 + 3x - 1 \quad \text{for} \quad x \neq 6$

- $f(x) = \begin{cases} x^2 + 3x - 1 & \text{for} \quad x \neq -1 \\ -9 & \text{for} \quad x = -1 \end{cases}$

- $f(x) = \begin{cases} x & \text{for} \quad x > 0 \\ -x^2 & \text{for} \quad x < 0 \end{cases}$

Funny functions might be continuous or discontinuous! Funny functions need to be tested to see if they are continuous or discontinuous.

The test

1. First ask yourself,"Where is my function doing something funny? At which value of x do I have division by zero, is my function not defined or do two separate parts of my function meet up?"
2. Then find the limit of your function as x approaches that value from the left.
3. Also find the limit of your function as x approaches that value from the right.
4. Find the value of your function at that value of x.

5. If your left limit, right limit and function value are all equal for that value of x then the function is continuous (unless there are funny things that are also happening at other points, in which case they also need to be tested). If they are not ALL equal then the function is discontinuous.

Let's work through three examples using this test so that you are comfortable with it. The first example is really obvious but it shows you how to use the test.

Example 1:

$$f(x) = \frac{x^2 - 5x + 6}{x - 3}$$

- This function is doing something funny at $x = 3$ because we have division by zero at $x = 3$.
- We find the limit as x approaches 3 from the left:

$$\begin{aligned} &\lim_{x \to 3^-} \frac{x^2 - 5x + 6}{x - 3} \\ = &\lim_{x \to 3^-} \frac{(x - 3)(x - 2)}{x - 3} \\ = &\lim_{x \to 3^-} (x - 2) \end{aligned}$$

(We have done our surgery so we can use direct substitution.)

$$\begin{aligned} =&(3 - 2) \\ =&1 \end{aligned}$$

- We find the limit as x approaches 3 from the right

$$\begin{aligned} &\lim_{x \to 3^+} \frac{x^2 - 5x + 6}{x - 3} \\ = &\lim_{x \to 3^+} \frac{(x - 3)(x - 2)}{x - 3} \\ = &\lim_{x \to 3^+} (x - 2) \\ =&(3 - 2) \\ =&1 \end{aligned}$$

- We find the value of our function at $x = 3$:

$$\begin{aligned} f(x) &= \frac{x^2 - 5x + 6}{x - 3} \\ \therefore f(3) &= \frac{(3)^2 - 5(3) + 6}{(3) - 3} \\ &= \frac{9 - 15 + 6}{0} \\ &= \frac{0}{0} \end{aligned}$$

Therefore the function value does not exist at $x = 3$.

The left limit is equal to the right limit but they are not equal to the function value at $x = 3$. Therefore

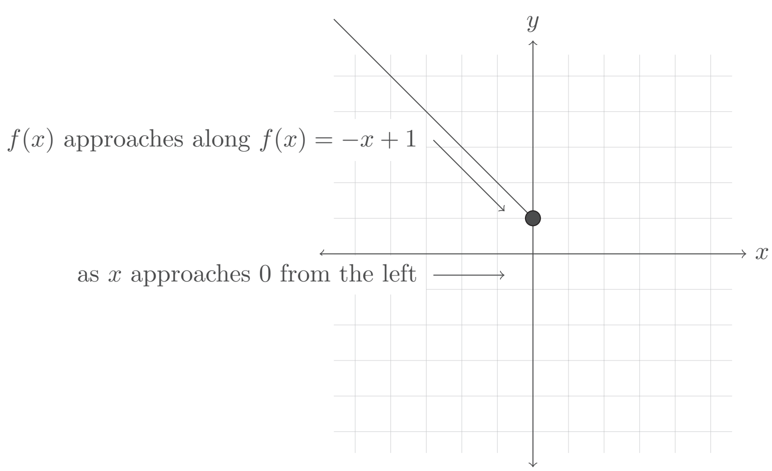
$$f(x) = \frac{x^2 - 5x + 6}{x - 3}$$

is not continuous.

Example 2:

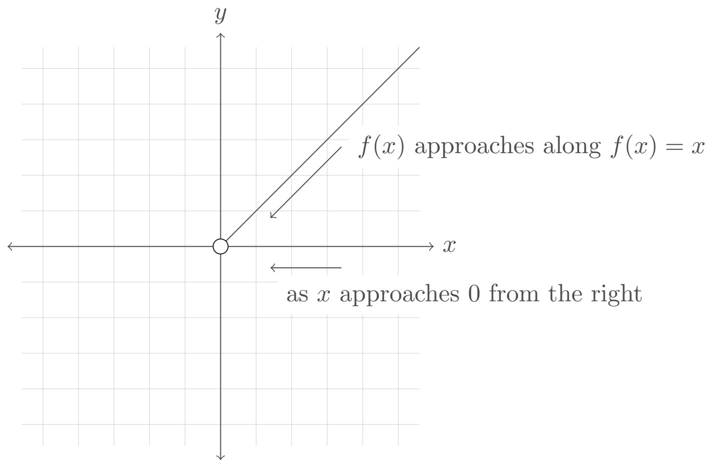
$$f(x) = \begin{cases} x & \text{for} \quad x > 0 \\ -x + 1 & \text{for} \quad x \leq 0 \end{cases}$$

- First we see that the function is doing something funny at $x = 0$ because that is where the separate parts join up.
- Then we find the left limit as x approaches 0. We can see that our function is defined by $f(x) = -x + 1$ when it is to the left of 0. Therefore, when x approaches 0 from the left, we are approaching along $f(x) = -x + 1$



$$\begin{aligned} &\therefore \lim_{x \to 0^-} f(x) \\ &= \lim_{x \to 0^-} (-x + 1) \\ &= 1 \end{aligned}$$

- Now we find the limit as our function approaches 0 from the right. We can see that our function is defined by $f(x) = x$ when it is to the right of 0. Therefore, when x approaches 0 from the right, we are approaching along $f(x) = x$.



$$
\begin{aligned}
&\therefore \lim_{x \to 0^+} f(x) \\
&= \lim_{x \to 0^+} x \\
&=0
\end{aligned}
$$

- Lastly, we find the value of our function at $x = 0$. We can see that at $x = 0$ our function is defined as $f(x) = -x + 1$. Therefore at $x = 0$:

$$
\begin{aligned}
f(x) &= -x + 1 \\
\therefore f(0) &= -(0) + 1 \\
&= 1
\end{aligned}
$$

- Now we compare all our values:

$$
\begin{aligned}
\lim_{x \to 0^-} f(x) &= 1 \\
\lim_{x \to 0^+} f(x) &= 0 \\
f(0) &= 1
\end{aligned}
$$

They are not all equal. Therefore

$$
f(x) = \begin{cases} x & \text{for} \quad x > 0 \\ -x + 1 & \text{for} \quad x \leq 0 \end{cases}
$$

is discontinuous.

This example was very detailed. We will go through the last example a lot quicker.

Example 3:

$$f(x) = \begin{cases} x & \text{for} \quad x > 0 \\ -x^2 & \text{for} \quad x \leq 0 \end{cases}$$

- The function is doing something funny at $x = 0$ because that is where the two separate parts meet.
- We find the left limit:

$$\begin{aligned} &\lim_{x \to 0^-} f(x) \\ =&\lim_{x \to 0^-} -x^2 \\ =&0 \end{aligned}$$

- We find the right limit:

$$\begin{aligned} &\lim_{x \to 0^+} f(x) \\ =&\lim_{x \to 0^+} x \\ =&0 \end{aligned}$$

- We find the function value at $x = 0$:

$f(x)$ is defined as $f(x) = -x^2$ when $x = 0$. Therefore:

$$\begin{aligned} f(0) &= -(0)^2 \\ &= 0 \end{aligned}$$

- Now we compare all our values:

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= 0 \\ \lim_{x \to 0^+} f(x) &= 0 \\ f(0) &= 0 \end{aligned}$$

They are all equal. Therefore

$$f(x) = \begin{cases} x & \text{for} \quad x > 0 \\ -x^2 & \text{for} \quad x \leq 0 \end{cases}$$

is continuous.

Question 3:

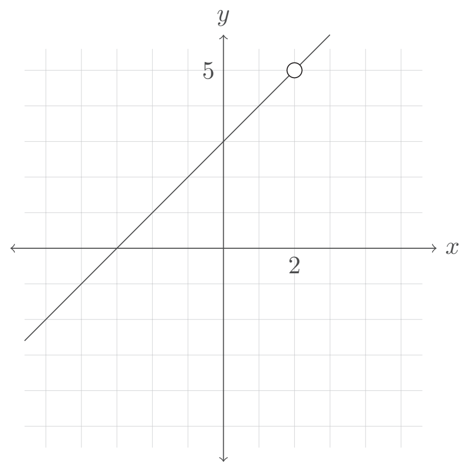
Determine whether the following functions are continuous or discontinuous:

	Is something funny happening and where?	Left limit	Right Limit	Function value	Conclusion
$f(x) = x$					
$f(x) = \dfrac{x^2 + 13x - 30}{x - 2}$					
$f(x) = \dfrac{5}{x - 3}$					
$f(x) = \begin{cases} -x + 3 & \text{for} \quad x \leq 4 \\ x + 5 & \text{for} \quad x > 4 \end{cases}$					
$f(x) = \begin{cases} \dfrac{x^2 + 13x - 30}{x - 2} & \text{for} \quad x \neq 2 \\ 16 & \text{for} \quad x = 2 \end{cases}$					
$f(x) = \begin{cases} \dfrac{x^2 + 13x - 30}{x - 2} & \text{for} \quad x \neq 2 \\ 17 & \text{for} \quad x = 2 \end{cases}$					
$f(x) = \begin{cases} x^3 & \text{for} \quad x < 0 \\ 5 & \text{for} \quad x = 0 \\ x^2 & \text{for} \quad x > 0 \end{cases}$					
$f(x) = \begin{cases} \dfrac{x^2 - 1}{x + 1} & \text{for} \quad x \neq -1 \\ -2 & \text{for} \quad x = -1 \end{cases}$					

Removable discontinuity

There is something special about a hole discontinuity. It is very easy to fix it. It is very easy to patch it up.

If we look at this graph, we can see the hole discontinuity at $x = 2$:



$$f(x) = \frac{x^2 + x - 6}{x - 2}$$

We can also see that the left limit and the right limit of our function, as x approaches 2, is 5. If you don't have a graph of your function, don't worry. You will know that you have a hole discontinuity if the left and right limits are equal but not equal to ∞. Look back at example 1; we know the function in that example has a hole discontinuity, even though we don't have a graph for it, because its left and right limits are equal.

So how do we patch up this hole discontinuity? It's this easy:

We just declare that, at $x = 2$, our function is equal to 17. So our new function will look like this:

$$f(x) = \begin{cases} \dfrac{x^2 + 13x - 30}{x - 2} & \text{for} \quad x \neq 2 \\ 17 & \text{for} \quad x = 2 \end{cases}$$

We defined a separate value for $f(x)$ at $x = 2$ that patches up the hole, and now we can see that:

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= 17 \\ \lim_{x \to 2^+} f(x) &= 17 \\ f(2) &= 17 \end{aligned}$$

So our function is now continuous!

Because it is so easy to patch up a hole discontinuity we also call it a removable discontinuity. We can remove the discontinuity from our function by simply defining a separate value for our function, at the value of x where the hole discontinuity occurs.

Question 4:

For the following functions, remove the discontinuity if possible:

	Is something funny happening and where?	Left limit	Right Limit	New function
$f(x) = \dfrac{x^2 + x}{x}$				
$f(x) = \dfrac{x^2 - 9}{x + 3}$				
$f(x) = \dfrac{1}{x - 2}$				
$f(x) = \dfrac{x^2 + 7x + 10}{x + 5}$				

Solutions

Solution 1:

In order from top to bottom:

Continuous

Continuous

Discontinuous (Gap)

Continuous

Discontinuous (Vertical asymptote)

Discontinuous (Hole)

Discontinuous (Hole)

Discontinuous (Gap)

Solution 2:

- $f(x) = \dfrac{x+1}{x}$

Division by 0 at $x = 0$.

- $f(x) = \dfrac{5}{x}$

Division by 0 at $x = 0$.

- $f(x) = \dfrac{x^2 - 5x + 6}{x - 3}$

Division by 0 at $x = 3$.

- $f(x) = \begin{cases} x & \text{for} \quad x < 0 \\ x + 1 & \text{for} \quad x \geq 0 \end{cases}$

Gap at $x = 0$.

- $f(x) = x^2 + 3x - 1 \quad \text{for} \quad x \neq 6$

Hole at $x = 6$.

- $f(x) = \begin{cases} x^2 + 3x - 1 & \text{for} \quad x \neq -1 \\ -9 & \text{for} \quad x = -1 \end{cases}$

Hole at $x = -1$.

- $f(x) = \begin{cases} x & \text{for} \quad x > 0 \\ -x^2 & \text{for} \quad x < 0 \end{cases}$

Hole at $x = 0$.

Solution 3:

	Is something funny happening and where?	Left limit	Right Limit	Function value	Conclusion
$f(x) = x$	No				C
$f(x) = \dfrac{x^2 + 13x - 30}{x - 2}$	Yes, at $x = 2$	17	17	d.n.e.	D
$f(x) = \dfrac{5}{x - 3}$	Yes, at $x = 3$	$-\infty$	∞	d.n.e.	D
$f(x) = \begin{cases} -x + 3 & \text{for} \quad x \leq 4 \\ x + 5 & \text{for} \quad x > 4 \end{cases}$	Yes, at $x = 4$	-1	9	-1	D
$f(x) = \begin{cases} \dfrac{x^2 + 13x - 30}{x - 2} & \text{for} \quad x \neq 2 \\ 16 & \text{for} \quad x = 2 \end{cases}$	Yes, at $x = 2$	17	17	16	D
$f(x) = \begin{cases} \dfrac{x^2 + 13x - 30}{x - 2} & \text{for} \quad x \neq 2 \\ 17 & \text{for} \quad x = 2 \end{cases}$	Yes, at $x = 2$	17	17	17	C
$f(x) = \begin{cases} x^3 & \text{for} \quad x < 0 \\ 5 & \text{for} \quad x = 0 \\ x^2 & \text{for} \quad x > 0 \end{cases}$	Yes, at $x = 0$	0	0	5	D
$f(x) = \begin{cases} \dfrac{x^2 - 1}{x + 1} & \text{for} \quad x \neq -1 \\ -2 & \text{for} \quad x = -1 \end{cases}$	Yes, at $x = -1$	-2	-2	-2	C

Solution 4:

	Is something funny happening and where?	Left limit	Right Limit	New function
$f(x) = \frac{x^2+x}{x}$	Yes, at $x=0$	1	1	$f(x) = \begin{cases} \frac{x^2+x}{x} & \text{for} \quad x \neq 0 \\ 1 & \text{for} \quad x = 0 \end{cases}$
$f(x) = \frac{x^2-9}{x+3}$	Yes, at $x=-3$	-6	-6	$f(x) = \begin{cases} \frac{x^2-9}{x+3} & \text{for} \quad x \neq -3 \\ -6 & \text{for} \quad x = -3 \end{cases}$
$f(x) = \frac{1}{x-2}$	Yes, at $x=2$	$-\infty$	∞	Discontinuity is a vertical asymptote so it can't be removed.
$f(x) = \frac{x^2+7x+10}{x+5}$	Yes, at $x=-5$	-3	-3	$f(x) = \begin{cases} \frac{x^2+7x+10}{x+5} & \text{for} \quad x \neq -5 \\ -3 & \text{for} \quad x = -5 \end{cases}$

Convergence and Power Series

Lesson 1: Convergence and convergence tests

Series

A series is an infinite number of terms being added together. We can't actually add all the terms of an infinit series together so we talk about the sum of the first n terms as n approaches infinity. If the sum of the first n terms approaches some finite value, as n approaches infinity, then we say the series is convergent. If the sum of the first n terms doesn't approach some finite value, as n approaches infinity, then we say the series is divergent.You have seen this before with geometric series. This is a geometric series:

$$a + ar + ar^2 + ar^3 + ...$$

If $|r| < 1$ then the series converges and its value is given by $\frac{a}{1-r}$. If $|r| \geq 1$ then the series diverges.

It's actually quite a strange idea; letting the number of terms in a series approach infinity but the series approaches a finite value! The story of the tortoise and Achilles, at the end of the lesson, might make you feel more comfortable with this strange idea.

In this lesson we will be looking at how to tell if a series is convergent. We have just seen that it is easy to tell if a geometric series is convergent but there are also other types of series for which it is not as obvious.

Harmonic series:

This series is possibly the most famous of all series and it is very simple:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$$

But does it converge or diverge? It diverges! I have given a very rough proof for this if you would like to read over it:

Let's start by adding a few of the terms at the beginning together:

$$1 + \frac{1}{2} = 1.5$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \left(\frac{1}{2}\right) = 2$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right) = 1+\frac{1}{2}+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right) = 2.5$$

And so it goes on. The pattern that can be seen here is that each time I want my series to increase by at least 0.5 I can add a finite number of terms of the series to get such an increase. Because the series has an infinite number of terms I will always have enough terms to make another increase of at least 0.5. Therefore the series must diverge.

Now it would be silly if we had to go through a proof like this every time we wanted to see if a series is convergent or divergent so I will show you some easy to use tests to use on series. Before we get to our first test we need to take a look at a definition:

Alternating series

An alternating series has alternating positive and negative terms. Here are two examples:

$$1 - 2 + 3 - 4 + 5 - 6 + ...$$
$$-1 + 2 - 3 + 4 - 5 + 6 - ...$$

The sum notation of the first series is:

$$\sum_{n=1}^{\infty} (-1)^{n-1} n$$

If this looks confusing, the following questions should clear things up:

Question 1: Write out the first 6 terms of the following series:

$$\sum_{n=1}^{\infty} n$$

Question 2:

Now compare your answer in question 1 to the first example above. How are they different?

Question 3:

Now compare:

$$\sum_{n=1}^{\infty} (-1)^{n-1} n \qquad \text{and} \qquad \sum_{n=1}^{\infty} n.$$

What is the $(-1)^{n-1}$ doing in the first sum notation?

Question 4:

Now try writing out the sum notation of the second example. You can't just use $(-1)^{n-1}$ again because then you will just get the first series. You will need to change $(-1)^{n-1}$ slightly to get the sum notation for the second series.

Question 5:

Write out the following two series using sum notation:

$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+...$	
$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+...$	

Question 6:

Try to deduce the name of the second series in question 5:

Alternating series test

This is how we can tell whether an alternating series converges:

If we have an alternating series:

$$a_1 - a_2 + a_3 - a_4 + ... \qquad (a_n > 0)$$

OR

$$-a_1 + a_2 - a_3 + a_4 - ... \qquad (a_n > 0)$$

And the following conditions are satisfied:

$$a_n \geq a_{n+1} \qquad \text{and} \qquad \lim_{n\to\infty} a_n = 0$$

Then the alternating series converges.

What this is saying is that if we look at the absolute value of each term, not the sign, and every number is smaller than or equal to the number just before it AND the numbers are approaching 0 as n approaches infinity, then the alternating series converges. If these conditions are not satisfied that doesn't mean that the alternating series is not converging, it just means that we can't say for sure that it is converging.

Question 7:

Are the following alternating series converging or can you not say?

$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+...$	
$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+...$	
$1\frac{1}{2}-1\frac{1}{3}+1\frac{1}{4}-1\frac{1}{5}+1\frac{1}{6}-...$	

Question 8:

So the alternating harmonic series is:

We know how to test if geometric and alternating series are convergent. To have a look at another test we need another definition.

Absolute convergence

If we take the absolute value of all the terms of a series, so everything becomes positive, and it still converges then it is absolutely convergent. Not much else to say on the matter, except this:

If a series is absolutely convergent then it is also convergent. Note: this does not mean that if a series is not absolutely convergent then it is not convergent. A series could be convergent but not be absolutely convergent. Read over this as many times as you need until you get the idea.

Question 9:

Can you think of a series that is convergent but not absolutely convergent?

Ratio test

And now for the best test! This is the best test because it can be used in many situations, so it is the most useful. I am going to use a geometric series as an example only because you are comfortable with it. Keep in mind that we don't actually have to use the ratio test to find out if a geometric series converges because we already have a test for that.

First we have to find the general term for series. Example for the series:

$$1+2+4+8+16+...$$

$$a=1$$
$$r=2$$

So the general term, T_n , is $ar^{n-1} = 1.2^{n-1} = 2^{n-1}$ where n starts at 1. If we are given the series in sum notation then we automatically get given the general term so we don't have to figure it out. If we were given the following, then we would automatically have the general term for our example:

$$\sum_{n=1}^{\infty} 2^{n-1}$$

Then we have to write out T_{n+1} , which means we have to add a 1 to all the n's in T_n:

$$\begin{aligned} T_n &= 2^{n-1} \\ \therefore T_{n+1} &= 2^{(n+1)-1} = 2^n \end{aligned}$$

We then put them into an absolute value ratio with T_{n+1} on top:

$$\left|\frac{T_{n+1}}{T_n}\right|$$

And in our example that would be:

$$\left|\frac{2^n}{2^{n-1}}\right|$$

The final step: we let n tend towards ∞ . In other words we find the limit as n goes to ∞, in the absolute value ratio.

$$\begin{aligned} &\lim_{n\to\infty}\left|\frac{T_{n+1}}{T_n}\right| \\ =&\lim_{n\to\infty}\left|\frac{2^n}{2^{n-1}}\right| \end{aligned}$$

Question 10:

Find this limit:

Once we have found the limit we need to know how to interpret it. Here's how:

- If $\lim_{n\to\infty}\left|\frac{T_{n+1}}{T_n}\right| < 1$ then the series converges absolutely! Which means it definitely converges.
- If $\lim_{n\to\infty}\left|\frac{T_{n+1}}{T_n}\right| > 1$ the series diverges.
- If $\lim_{n\to\infty}\left|\frac{T_{n+1}}{T_n}\right| = 1$ then the series could converge OR diverge, we don't know.

The limit of our example was 2 so $1 + 2 + 4 + 8 + 16 + ...$ must diverge... but we already knew that because it's a geometric series and $r > 1$. So let's try a few examples where we *have to* use the ratio test to find out if the series converges or diverges.

Question 11:

Use the ratio test to determine if the following series converge or diverge:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+... \qquad \text{(another geometric series, just to get the feel of it)}$$

$$\sum_{n=1}^{\infty}\frac{n^2}{2^n}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3}$$

The harmonic series

Zeno's Paradox of the Tortoise and Achilles

Zeno of Elea (circa 450 b.c.) is credited with creating several famous paradoxes, but by far the best known is the paradox of the Tortoise and Achilles. (Achilles was the great Greek hero of Homer's The Iliad.) It has inspired many writers and thinkers through the ages, notably Lewis Carroll and Douglas Hofstadter, who also wrote dialogues involving the Tortoise and Achilles. The original goes something like this:

The Tortoise challenged Achilles to a race, claiming that he would win as long as Achilles gave him a small head start. Achilles laughed at this, for of course he was a mighty warrior and swift of foot, whereas the Tortoise was heavy and slow. "How big a head start do you need?" he asked the Tortoise with a smile. "Ten meters," the latter replied. Achilles laughed louder than ever. "You will surely lose, my friend, in that case," he told the Tortoise, "but let us race, if you wish it." "On the contrary," said the Tortoise, "I will win, and I can prove it to you by a simple argument." "Go on then," Achilles replied, with less confidence than he felt before. He knew he was the superior athlete, but he also knew the Tortoise had the sharper wits, and he had lost many a bewildering argument with him before this. "Suppose," began the Tortoise, "that you give me a 10-meter head start. Would you say that you could cover that 10 meters between us very quickly?" "Very quickly," Achilles affirmed. "And in that time, how far should I have gone, do you think?" "Perhaps a meter - no more," said Achilles after a moment's thought. "Very well," replied the Tortoise, "so now there is a meter between us. And you would catch up that distance very quickly?" "Very quickly indeed!" "And yet, in that time I shall have gone a little way farther, so that now you must catch that distance up, yes?"

"Ye-es," said Achilles slowly. "And while you are doing so, I shall have gone a little way farther, so that you must then catch up the new distance," the Tortoise continued smoothly. Achilles said nothing. "And so you see, in each moment you must be catching up the distance between us, and yet I - at the same time - will be adding a new distance, however small, for you to catch up again." "Indeed, it must be so," said Achilles wearily. "And so you can never catch up," the Tortoise concluded sympathetically. "You are right, as always," said Achilles sadly - and conceded the race. Zeno's Paradox may be rephrased as follows. Suppose I wish to cross the room. First, of course, I must cover half the distance. Then, I must cover half the remaining distance. Then, I must cover half the remaining distance. Then I must cover half the remaining distance . . . and so on forever. The consequence is that I can never get to the other side of the room. What this actually does is to make all motion impossible, for before I can cover half the distance I must cover half of half the distance, and before I can do that I must cover half of half of half of the distance, and so on, so that in reality I can never move any distance at all, because doing so involves moving an infinite number of small intermediate distances first. Now, since motion obviously is possible, the question arises, what is wrong with Zeno? What is the "flaw in the logic?" If you are giving the matter your full attention, it should begin to make you squirm a bit, for on its face the logic of the situation seems unassailable. You shouldn't be able to cross the room, and the Tortoise should win the race! Yet we know better. Hmm. Rather than tackle Zeno head-on, let us pause to notice something remarkable. Suppose we take Zeno's Paradox at face value for the moment, and agree with him that before I can walk a mile I must first walk a half-mile. And before I can walk the remaining half-mile I must first cover half of it, that is, a quarter-mile, and then an eighth-mile, and then a sixteenth-mile, and then a thirty-secondth-mile, and so on. Well, suppose I could cover all these infinite number of small distances, how far should I have walked? One mile! In other words,

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ...$$

At first this may seem impossible: adding up an infinite number of positive distances should give an infinite distance for the sum. But it doesn't - in this case it gives a finite sum; indeed, all these distances add up to 1! A little reflection will reveal that this isn't so strange after all: if I can divide up a finite distance into an infinite number of small distances, then adding all those distances together should just give me back the finite distance I started with. (An infinite sum such as the one above is known in mathematics as an infinite series, and when such a sum adds up to a finite number we say that the series is convergent.) Now the resolution to Zeno's Paradox is easy. Obviously, it will take me some fixed time to cross half the distance to the other side of the room, say 2 seconds. How long will it take to cross half the remaining distance? Half as long - only 1 second. Covering half of the remaining distance (an eighth of the total) will take only half a second. And so on. And once I have covered all the infinitely many sub-distances and added up all the time it took to traverse them? Only 4 seconds, and here I am, on the other side of the room after all. And poor old Achilles would have won his race.

Solutions

Solution 1:

1+2+3+4+5+6+...

Solution 2:

The sign of each term is alternating between + and - in the first alternating series example.

Solution 3:

The $(-1)^{n-1}$ part of the sum notation causes the sign of each term to alternate:

$$\begin{array}{lll} \text{For} & n=1, & (-1)^{n-1}=1 \\ \text{For} & n=2, & (-1)^{n-1}=-1 \\ \text{For} & n=3, & (-1)^{n-1}=1 \\ \text{For} & n=4, & (-1)^{n-1}=-1 \\ \text{For} & n=5, & (-1)^{n-1}=1 \end{array}$$

And so on...

Solution 4:

$$\sum_{n=1}^{\infty}(-1)^n n \quad \text{is the sum notation for the second alternating series example.}$$

$$\begin{array}{lll} \text{For} & n=1, & (-1)^{n}=-1 \\ \text{For} & n=2, & (-1)^{n}=1 \\ \text{For} & n=3, & (-1)^{n}=-1 \\ \text{For} & n=4, & (-1)^{n}=1 \\ \text{For} & n=5, & (-1)^{n}=-1 \end{array}$$

And so on...

Solution 5:

$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+...$	$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}$
$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+...$	$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

Solution 6:

The second series, in question 5, is called the *alternating harmonic series*.

Solution 7:

$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+...$	$\lim_{n\to\infty} a_n \neq 0$ and $a_n \leq a_{n+1}$ so we can't tell if it is converging.
$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+...$	$\lim_{n\to\infty} a_n = 0$ and $a_n \geq a_{n+1}$ so the series is converging.
$1\frac{1}{2}-1\frac{1}{3}+1\frac{1}{4}-1\frac{1}{5}+1\frac{1}{6}-...$	$a_n \geq a_{n+1}$ but $\lim_{n\to\infty} a_n \neq 0$ so we can't tell if it is converging.

Solution 8:

So the alternating harmonic series is: *Convergent*.

Solution 9:

The alternating harmonic series.

Solution 10:

$$
\begin{aligned}
&= \lim_{n\to\infty}\left|\frac{2^n}{2^{n-1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{2^n}{2^n.2^{-1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{2^{-1}}\right| \\
&= \lim_{n\to\infty} 2 \\
&=2
\end{aligned}
$$

Solution 11:

$$\boxed{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...}$$

$$
\begin{aligned}
&\lim_{n\to\infty}\left|\frac{T_{n+1}}{T_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{1}{2^n.2^1}}{\frac{1}{2^n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{2^1}\right| \\
&=\frac{1}{2}
\end{aligned}
$$

$\therefore$ this series converges

$$\boxed{\sum_{n=1}^{\infty}\frac{n^2}{2^n}}$$

$$
\begin{aligned}
&\lim_{n\to\infty}\left|\frac{T_{n+1}}{T_n}\right| \\
=&\lim_{n\to\infty}\left|\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}\right| \\
=&\lim_{n\to\infty}\left|\frac{\frac{(n+1)^2}{2^n.2^1}}{\frac{n^2}{2^n}}\right| \\
=&\lim_{n\to\infty}\left|\frac{\frac{(n+1)^2}{2^1}}{n^2}\right| \\
=&\lim_{n\to\infty}\left|\frac{n^2+2n+1}{2n^2}\right| && \text{(divide top and bottom by } n^2\text{)} \\
=&\lim_{n\to\infty}\left|\frac{1+\frac{2}{n}+\frac{1}{n^2}}{2}\right| \\
=&\frac{1}{2}
\end{aligned}
$$

$\therefore$ this series converges

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^3}}$$

$$
\begin{aligned}
&\lim_{n\to\infty}\left|\frac{T_{n+1}}{T_n}\right| \\
=&\lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}}\right| \\
=&\lim_{n\to\infty}\left|\frac{n^3}{(n+1)^3}\right| \\
=&\lim_{n\to\infty}\left|\frac{n^3}{n^3+3n^2+3n+1}\right| && \text{(divide top and bottom by } n^3\text{)} \\
=&\lim_{n\to\infty}\left|\frac{1}{1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}}\right| \\
=&1
\end{aligned}
$$

$\therefore$ ratio test is inconclusive

$\boxed{\text{The harmonic series.}}$

$$
\begin{aligned}
&\lim_{n\to\infty}\left|\frac{T_{n+1}}{T_n}\right| \\
=&\lim_{n\to\infty}\left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| \\
=&\lim_{n\to\infty}\left|\frac{n}{n+1}\right| \qquad\qquad \text{(divide top and bottom by } n\text{)} \\
=&\lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| \\
=&1
\end{aligned}
$$

$\therefore$ ratio test is inconclusive

Convergence and Power Series

Lesson 2: Power series

Polynomials

$$\begin{array}{ll} \text{What is this:} & x^2 + 5 \\ \text{Or this:} & x^3 - 2x^2 + x - 1 \end{array}$$

These expressions, and others like them, are called polynomials. When we talk about the degree of a polynomial we are talking about the value of the highest value exponent. So the degree of the first polynomial is 2 and the degree of the second polynomial is 3. A polynomial can have any degree, degree 1 000 000 if you want, but that's really big! We can even use sum notation to describe any polynomial (think about it, a polynomial is just a finite series). So the first polynomial above could be written like this:

$$x^2 + 5 = \sum_{n=0}^{2} c_n x^n \qquad \text{where} \qquad \begin{array}{l} c_0 = 5 \\ c_1 = 0 \\ c_2 = 1 \end{array}$$

(Note how the terms are written out in reverse. The constant is the first term of the sum form but the last term of the usual form.)

Question 1:

Study the first example and then try do the second one yourself.

$$x^3 - 2x^2 + x - 1 =$$

$$\text{Where} \begin{array}{l} c_0 = \_\_\_\_ \\ c_1 = \_\_\_\_ \\ c_2 = \_\_\_\_ \\ c_3 = \_\_\_\_ \end{array}$$

However, the frustrating thing is that, if we are writing any random polynomial in sum notation, we have to list all the c_n's (the coefficients of the polynomial).

It's much easier if we go in the opposite direction, if we are given the sum notation of the polynomial first and the coefficients are given as constants or described in terms of n. Example:

$$\sum_{n=0}^{4} 3x^n = 3 + 3x + 3x^2 + 3x^3 + 3x^4$$

Here, every c_n is 3 because that is how it is described in sum notation.

$$\sum_{n=0}^{3} nx^n = x + 2x^2 + 3x^3$$

Here the value of each c_n is equal to n, so it has a pattern that is easy to describe in sum notation.

Question 2:

Write out the following polynomials in their expanded form:

$\sum_{n=0}^{5} x^n$	
$\sum_{n=0}^{3} \left(\frac{1}{1+n}\right) x^n$	

Power series

A power series can be thought of as an infinite polynomial, it just keeps going forever. Example:

$$1 + x + x^2 + x^3 + x^4 + ...$$

Question 3:

Write out the above power series in sum notation:

____

Question 4:

Write out the following using sum notation:

$5 + 5x + 5x^2 + 5x^3 + ...$	
$1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + ...$	

Power series as a function

We know that any polynomial can be plotted on a graph and if you substitute a value for x, you will get some value out. Substitute 4 in to $1+x+x^2$ and you get 21 out. So why should a power series be any different? It's not. You can think of a power series as a function in much the same way as you thought of a polynomial as a function. However, can you think of a potential problem that might arise with a power series as a function?

Because a power series has an infinite number of terms there is the possibility that all the terms could add up to infinity, for certain values of x. Now we also know, from the previous lesson, that it is very possible to get a finite value when we add up an infinite number of terms. So the question becomes: which values of x will make a power series equal to infinity and which values for x will make a power series add up to a finite value? In other words: which values for x will make a power series diverge or converge?

If we think of a power series as a function then a power series is only defined for the values of x that make it converge. At other values of x, the series diverges and the function is undefined.

Now we are going to use the ratio test to determine convergence of a power series. However, there is a big difference between using the ratio test on a series and a power series. The ratio test for a series is either less than, equal to or greater than 1. We can immediately say whether it converges, diverges or is inconclusive. Take a look at what happens when we use the ratio test on a power series:

A simple power series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$$

To apply the ratio test to this series, we write:

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right|$$
$$= |x|$$

We are using the same old ratio test but now there are variables involved.

This is the answer, but is it less than, equal to or greater than 1? Well, it all depends what x is equal to. If $-1 < x < 1$ then clearly the ratio test is less than 1 and if the ratio test is less than 1 then the power series converges. So we say:

The power series converges for $-1 < x < 1$. So if we were to draw the power series as a graph we would only be able to draw the graph between -1 and 1 on the x axis. Don't worry about the values of x that make the ratio test equal to 1, you will have plenty of time to think about those values at university.

Question 5:

What is the difference between a series and a power series?

Interval of convergence

$-1 < x < 1$ is called the **interval of convergence**. We can also talk about the **radius of convergence**, which is equal to half the length of the interval of convergence.

Question 6:

Give the interval of convergence and the radius of convergence of:

$$\sum_{n=0}^{\infty} x^n$$

(The power series we have just done)

Interval of convergence	
Radius of convergence	

Question 7:

Now find the interval of convergence and radius of convergence of the following power series:

	Interval of convergence	Radius of convergence
$\sum_{n=1}^{\infty} \frac{x^n}{n}$ (n must start at 1, if it started at 0 the first term would be undefined)		
$\sum_{n=0}^{\infty} \frac{3^n x^n}{(n+1)^2}$		

There is one last point to make before we look at what these power series are good for.

Question 8:

What have you noticed about the position of the intervals of convergence on the x-axis?

The center of the interval of convergence can be shifted around by adding or subtracting a constant to or from all the x's in the power series. Let's use the following power series as an example:

$$\sum_{n=0}^{\infty} x^n$$

It's interval of convergence is $-1 < x < 1$, whereas:

$$\sum_{n=0}^{\infty} (x-1)^n$$

has an interval of convergence of $0 < x < 2$. We use the same shifting rules as you have used for shifting graphs left or right, by adding numbers to or subtracting numbers from x.

Question 9:

If I rewrite:

$$\sum_{n=0}^{\infty} x^n$$

as

$$\sum_{n=0}^{\infty}(x+1)^n \quad \text{or} \quad \sum_{n=0}^{\infty}(x-2)^n$$

what do the new intervals become? (Hint: Don't do any calculation. This is a common sense question.)

	New interval of convergence
$\sum_{n=0}^{\infty}(x+1)^n$	
$\sum_{n=0}^{\infty}(x-2)^n$	

A power series written like this is said to be **about** 'the constant with opposite sign'. Example:

$$\sum_{n=0}^{\infty}(x+1)^n \quad \text{is } \textbf{about} \text{ -1}$$

$$\sum_{n=0}^{\infty}(x-2)^n \quad \text{is } \textbf{about} \text{ 2}$$

Question 10:

Find the interval of convergence of the following: (Hint: look at the interval of convergence of the first power series in question 6)

$\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}$	

Can you see how similar this type of shifting is to how you shift a function from left to right on a Cartesian plane? It's actually the exact same thing because, at the end of the day, a power series is a function!

What are power series good for?

Power series can be used to represent functions in a different way. For example:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + ...$$

It may seem pretty strange now but you will find out more about how this works at university.

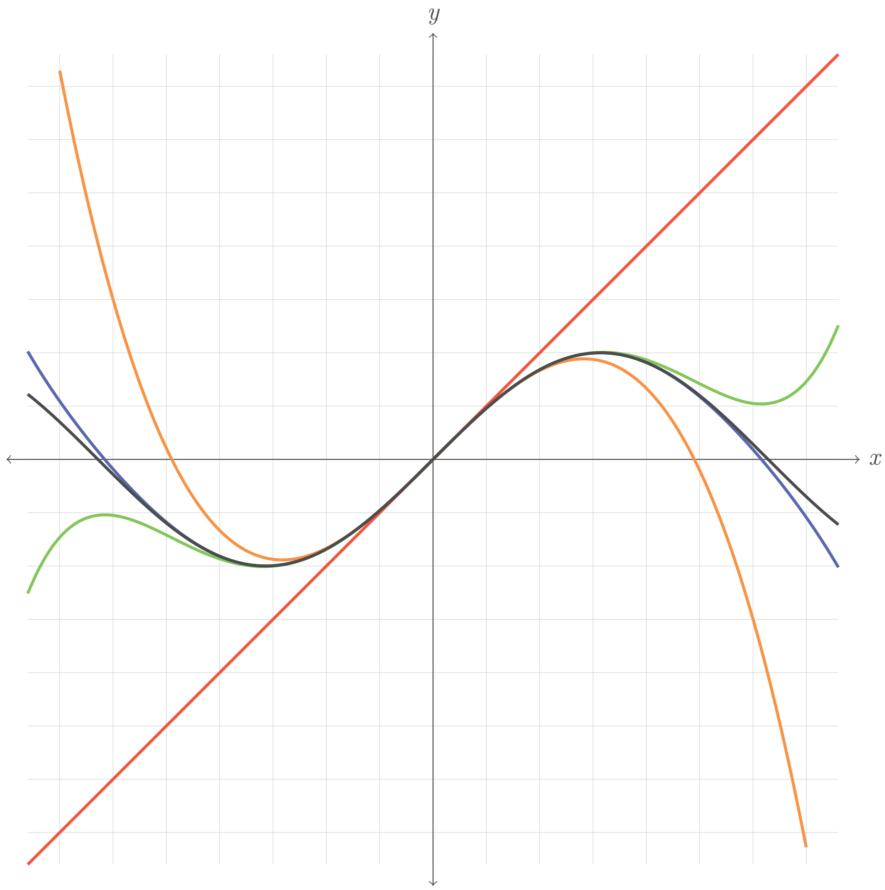
If we don't write all the terms of the series to infinity then we will only have an approximation to the function. Example:

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

OR

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

These approximations are themselves functions and we can see on the graph below that the more terms they have, the more they resemble $\sin x$.



- $\sin x$
- x
- $x - \frac{x^3}{3!}$
- $x - \frac{x^3}{3!} + \frac{x^5}{5!}$
- $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

And if we use an infinite number of terms, the series is no longer an approximation but is equal to the function that it represents.

Question 11:

Which constant are these approximations **about**?

The constant which these approximations are **about** will be the position on the x-axis where they best approximate the function. Example:

$$(x+4) - \frac{(x+4)^3}{3!} + \frac{(x+4)^5}{5!} - \frac{(x+4)^7}{7!}$$

best approximates $\sin x$ near $x = -4$ on the graph. We can see that the approximations on the previous page are **about** 0 and if you look at the graphs of the approximations you will see that they are very good approximations near 0 on the x-axis.

Some terminology

We have already seen that, if we use a finite number of terms to represent a function, we call it an **approximation** of the function.

If we use a power series (which means it has an infinite number of terms) to represent a function and that power series is equal to the function then we call it a **power series expansion** of the function.

If a power series expansion of a function is **about** 0 then it is called a **Maclaurin series**.

If a power series expansion of a function is **about** any other constant then it is called a **Taylor series**.

There is a method for writing out any function as a power series expansion but that is a surprise best left for university.

Solutions

Solution 1:

$$x^3 - 2x^2 + x - 1 = \sum_{n=0}^{3} c_n x^n$$

$$\text{Where} \begin{array}{l} c_0 = -1 \\ c_1 = 1 \\ c_2 = -2 \\ c_3 = 1 \end{array}$$

Solution 2:

$\sum_{n=0}^{5} x^n$	$x^5 + x^4 + x^3 + x^2 + x + 1$
$\sum_{n=0}^{3} \left(\frac{1}{1+n}\right) x^n$	$\frac{1}{4}x^3 + \frac{1}{3}x^2 + \frac{1}{2}x + 1$

Solution 3:

$$\sum_{n=0}^{\infty} x^n$$

Solution 4:

$5 + 5x + 5x^2 + 5x^3 + \ldots$	$\sum_{n=0}^{\infty} 5x^n$
$1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \ldots$	$\sum_{n=0}^{\infty} \frac{1}{1+n} x^n$

Solution 5:

Each term of a normal series is a constant and the series either converges or diverges.

Each term of a power series contains a variable and the power series can converge or diverge, depending on which value you make the variable.

Solution 6:

Interval of convergence	$-1 < x < 1$
Radius of convergence	1

Solution 7:

	Interval of convergence	Radius of convergence
$\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n}$	$\displaystyle\lim_{n\to\infty}\left\lvert\frac{T_{n+1}}{T_n}\right\rvert$ $\displaystyle=\lim_{n\to\infty}\left\lvert\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right\rvert$ $\displaystyle=\lim_{n\to\infty}\left\lvert\frac{\frac{x^n.x^1}{n+1}}{\frac{x^n}{n}}\right\rvert$ $\displaystyle=\lim_{n\to\infty}\left\lvert\frac{\frac{x}{n+1}}{\frac{1}{n}}\right\rvert$ $\displaystyle=\lim_{n\to\infty}\left\lvert\frac{nx}{n+1}\right\rvert$ $\displaystyle=\lim_{n\to\infty}\left\lvert\frac{x}{1+\frac{1}{n}}\right\rvert$ $=\lvert x\rvert$ $\therefore$interval of convergence is: $-1<x<1$	1
$\displaystyle\sum_{n=0}^{\infty}\frac{3^n x^n}{(n+1)^2}$	$\displaystyle\lim_{n\to\infty}\left\lvert\frac{T_{n+1}}{T_n}\right\rvert$ $\displaystyle=\lim_{n\to\infty}\left\lvert\frac{\frac{3^{n+1}x^{n+1}}{(n+2)^2}}{\frac{3^n x^n}{(n+1)^2}}\right\rvert$ $\displaystyle=\lim_{n\to\infty}\left\lvert\frac{\frac{3^n.3^1x^n.x^1}{n^2+4n+4}}{\frac{3^n x^n}{n^2+2n+1}}\right\rvert$ $\displaystyle=\lim_{n\to\infty}\left\lvert\frac{\frac{3x}{n^2+4n+4}}{\frac{1}{n^2+2n+1}}\right\rvert$ $\displaystyle=\lim_{n\to\infty}\left\lvert\frac{3x.n^2+6xn+3x}{n^2+4n+4}\right\rvert$ $\displaystyle=\lim_{n\to\infty}\left\lvert\frac{3x+\frac{6x}{n}+\frac{3x}{n^2}}{1+\frac{4}{n}+\frac{4}{n^2}}\right\rvert$ $=\lvert 3x\rvert$ $\therefore$interval of convergence is: $\displaystyle-\frac{1}{3}<x<\frac{1}{3}$	$\frac{1}{3}$

Solution 8:

They are centered around 0

Solution 9:

	New interval of convergence
$\sum_{n=0}^{\infty}(x+1)^n$	$-2 < x < 0$
$\sum_{n=0}^{\infty}(x-2)^n$	$1 < x < 3$

Solution 10:

$\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}$	$2 < x < 4$

Solution 11:

0

Differentiation

Lesson 1: Rates of change

Change

It's all around us. Nothing in this world stays the same. There are many things that may not change within your lifetime but rest assured, eventually they will.

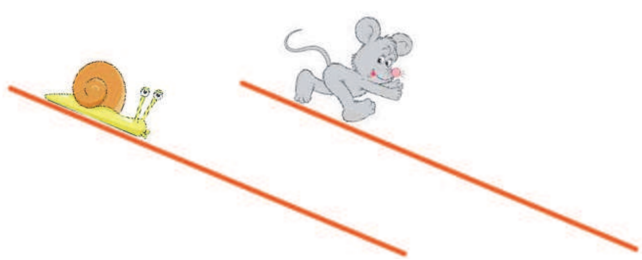
We grasp the concept of change from a very early age. Something is a certain way and then, given some amount of time, it is another way. Something can change colour, shape or position but, however it changes, it needs time to do so.

The idea of change is not as simple as: "Is something changing or not?" There is another very important idea that we are going to look at:

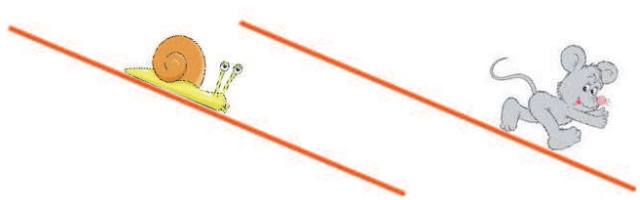
How fast does something change?

Rate of change

I have two pieces of string. One has a snail on it and the other has a mouse on it:



They are both moving along their pieces of string. Another way of thinking about this situation: They are both **changing** their positions along their pieces of string. Here are their postions 2 seconds later:



It is clear that the mouse's position is changing faster than the snail's position. This is the idea about change that we are interested in; how fast is something changing. But we are not happy with simply saying: "The mouse's position is changing quickly and the snail's position is changing slowly." We want to be precise; we want to be mathematical about this situation.

Let's assume that the mouse moved 4cm and the snail 2cm, during the 2 seconds.

1. We ask ourselves, "What is changing?"

   Answer:

   Position.

2. We ask ourselves, "By how much has it changed?"

   Answer:

   Mouse has changed its position by 4cm

   Snail has changed its position by 2cm

3. We ask ourselves, "How much time did it take for that thing to change by that amount?"

   Answer:

   Mouse took 2 seconds.

   Snail took 2 seconds.

4. We divide the amount by which our thing changed by the amount of time it took to change.

   Answer:

   Mouse, $\frac{4cm}{2sec} = 2cm/sec$

   Snail, $\frac{2cm}{2sec} = 1cm/sec$

These are measurements of how fast their positions are changing. These are called rates of change.

A more general way of writing this would be:

$$\frac{\text{Change in Postion}}{\text{Change in Time}}$$

In mathematics, the symbol for 'change in' is Δ.

So we have:

$$\frac{\Delta\text{Postion}}{\Delta\text{Time}}$$

This formula will give you the **rate of change** of position. 'Rate of change of position' is also known as 'velocity' but we will stick with 'Rate of change of position' for this section because we are interested in the general concept of a rate of change and don't want to confine our thinking to a specific type of rate of change.

Question 1:

This bucket started with $50ml$ of water in it. 150 seconds later it had $75ml$ in it. How fast is the volume changing? (give the answer in the same units)

Question 2:

This bucket started with $200cm^3$ of water in it. 3 minutes later it had $1400cm^3$ in it. What is the rate of change of the volume of water in this bucket? (give the answer in the same units)

Question 3:

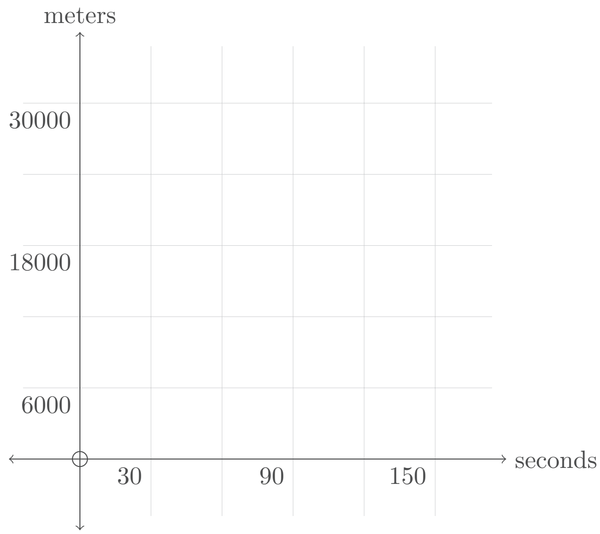
This bucket started with $1l$ of water in it. 5 seconds later it had $6l$ of water in it. Calculate $\frac{\Delta\text{Volume}}{\Delta\text{Time}}$.(give the answer in the same units)

Question 4:

A jet flies over Cape Town. Chris saw the jet fly over Fish Hoek at 13:00:00. Sipamandla saw the jet fly over Khayelitsha at 13:02:30 (150 seconds later). If Khayelitsha is $30km$ from Fish Hoek how fast is the jet flying in meters/second?

Question 5:

Sketch the graph of the distance flown by the jet, against time, on the set of axes below:



Question 6:

Find the gradient of the graph in question 5 and compare it to the rate of change you worked out in question 4:

Rate of change	Gradient

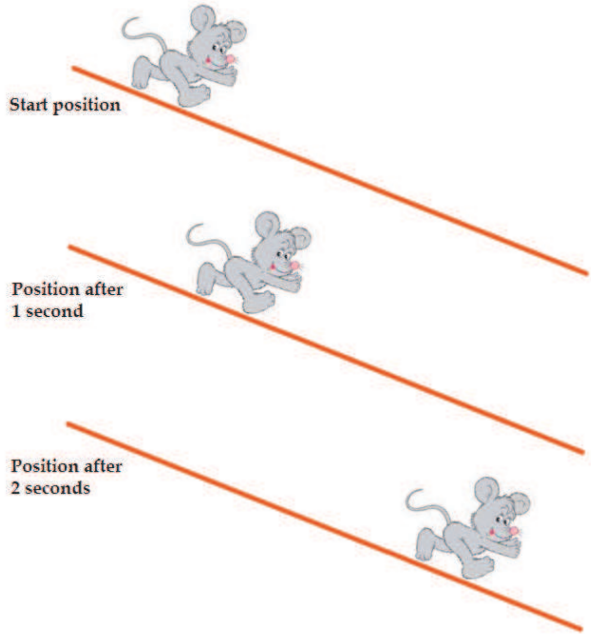
The rate of change is equal to the gradient. Rates of change and gradients are the same thing!

The jet has a constant rate of change. This rate of change is a number that we only need to calculate once. The graph for any example with a constant rate of change is a straight line and, as we have just seen, the gradient of the straight line is equal to the rate of change. We will now move onto the next idea.

Average rate of change

As was mentioned at the beginning of the lesson, "everything changes eventually". This is even true for rates of change themselves! Even rates of change, change.

Let's use our mouse on a string as an example:



During the 1st second our mouse moves $1cm$ but during the 2nd second he moves $3cm$. The position of our mouse is not only changing, it is changing faster and FASTER as time goes by.

So what is his rate of change of position? Let's try and work it out:

$$\text{Rate of change} = \frac{\Delta\text{Postion}}{\Delta\text{Time}}$$

For the 1st second we have:

$$\frac{\Delta\text{Postion}}{\Delta\text{Time}} = \frac{1}{1} = 1cm/sec$$

For the 2nd second we have:

$$\frac{\Delta\text{Postion}}{\Delta\text{Time}} = \frac{3}{1} = 3cm/sec$$

The mouse has a different rate of change of position over different time intervals. There is no one value for the rate of change (like there was for the jet), the rate of change is changing all the time! At each instant in time the rate of change is different.

Be careful not to make the mistake of thinking that the mouse is moving at $1cm/sec$ for the whole of the 1st second or that he is moving at $3cm/sec$ for the whole of the 2nd second. At each instant in time between 0 and 1 second he has a different rate of change and it is the average of all these instant rates of change that equals $1cm/sec$. At each instant in time between 1 and 2 seconds he also has a different rate of change and it is the average of all these instantaneous rates of change that equals $3cm/sec$.

In short, if a rate of change is not constant then we calculate the **average rate of change** for a given time interval.

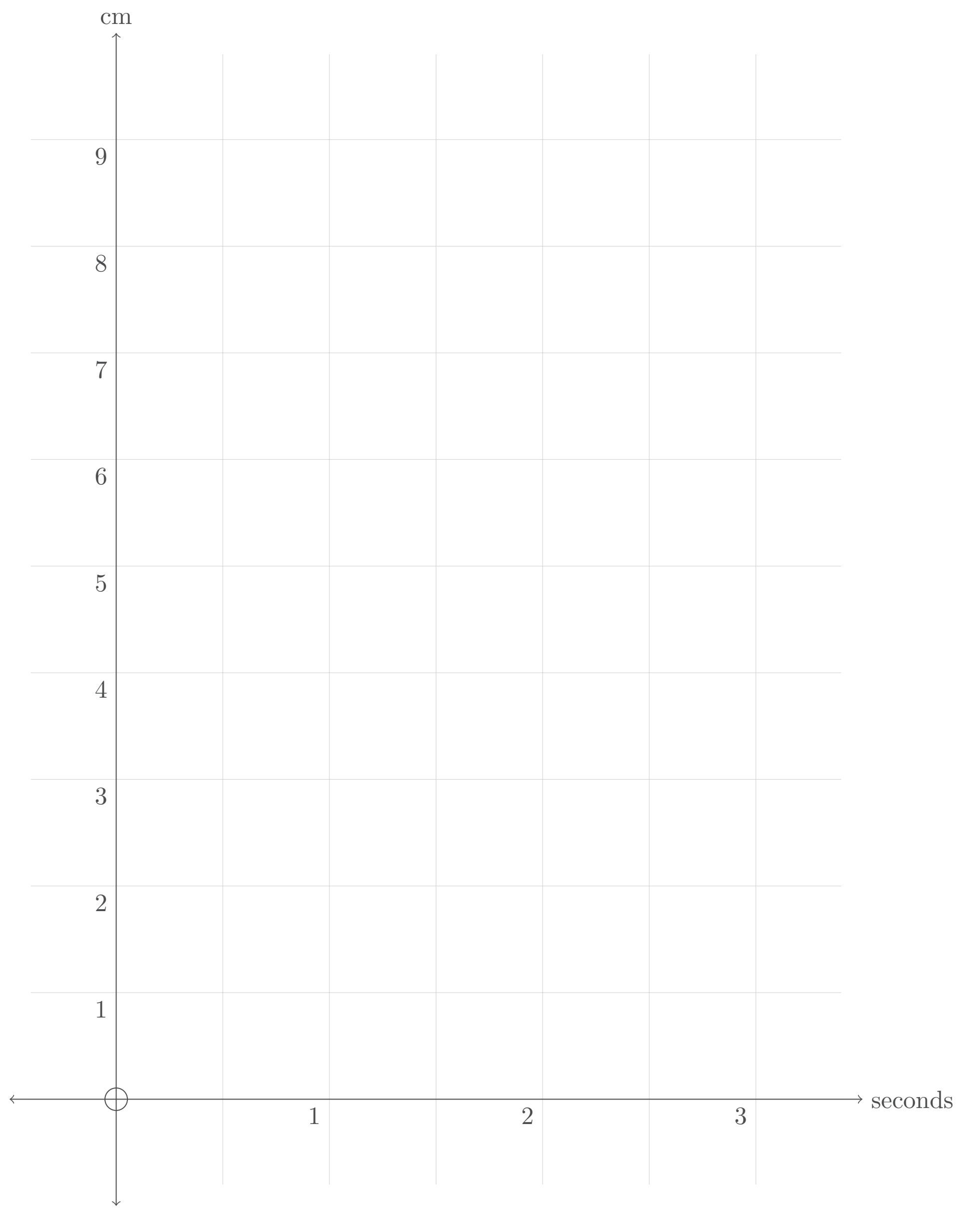
Question 7:

What is the average rate of change of the mouse's position between 0 and 2 seconds?

Question 8:

Now plot the positions of the mouse at different times, on the set of axes that follows and sketch the curve that runs through the points: (Take note of the different scales for the two axes. 1 block = 1 cm whereas 2 blocks = 1 second.)

Time (sec)	Position (cm)
1	1
2	4
3	9



Notice that the curve you have sketched doesn't have a gradient. If we want to find the average rate of change between two points then we have to draw a straight line between those two points and find its gradient.

Question 9:

Draw a straight line that cuts through the curve you have just drawn, at time = 1 second and time = 3 seconds and find its gradient.

The gradient you have just calculated is the average rate of change of the mouse's position between 1 and 3 seconds. Once again we see that a rate of change, even though it is an average rate of change, is represented by a gradient.

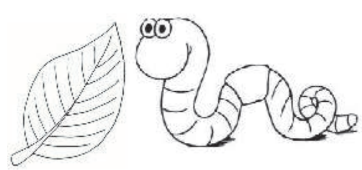
Question 10:

Find the average rate of change of the mouse's postion, for the following time intervals and draw the straight lines that represent those average rates of change on the graph that you sketched in question 8:

Time interval	Average rate of change
0 to 3 seconds	
2 to 3 seconds	

Question 11:

This worm is eating a leaf!



The area of the leaf is $10cm^2$. The worm is very hungry and starts out by eating very fast. As he eats more leaf he fills up and his eating slows down. Overleaf is a table of the total amount of leaf that he has eaten at each time:

Time (seconds)	Total amount of leaf eaten (cm^2)
0	0
1	4
2	7
3	9
4	10

Find the average rate of change of the amount of leaf eaten by the worm, for the following time intervals:

Time interval	Average rate of change
0 to 4 seconds	
0 to 1 second	
1 to 3 seconds	
2 to 4 seconds	

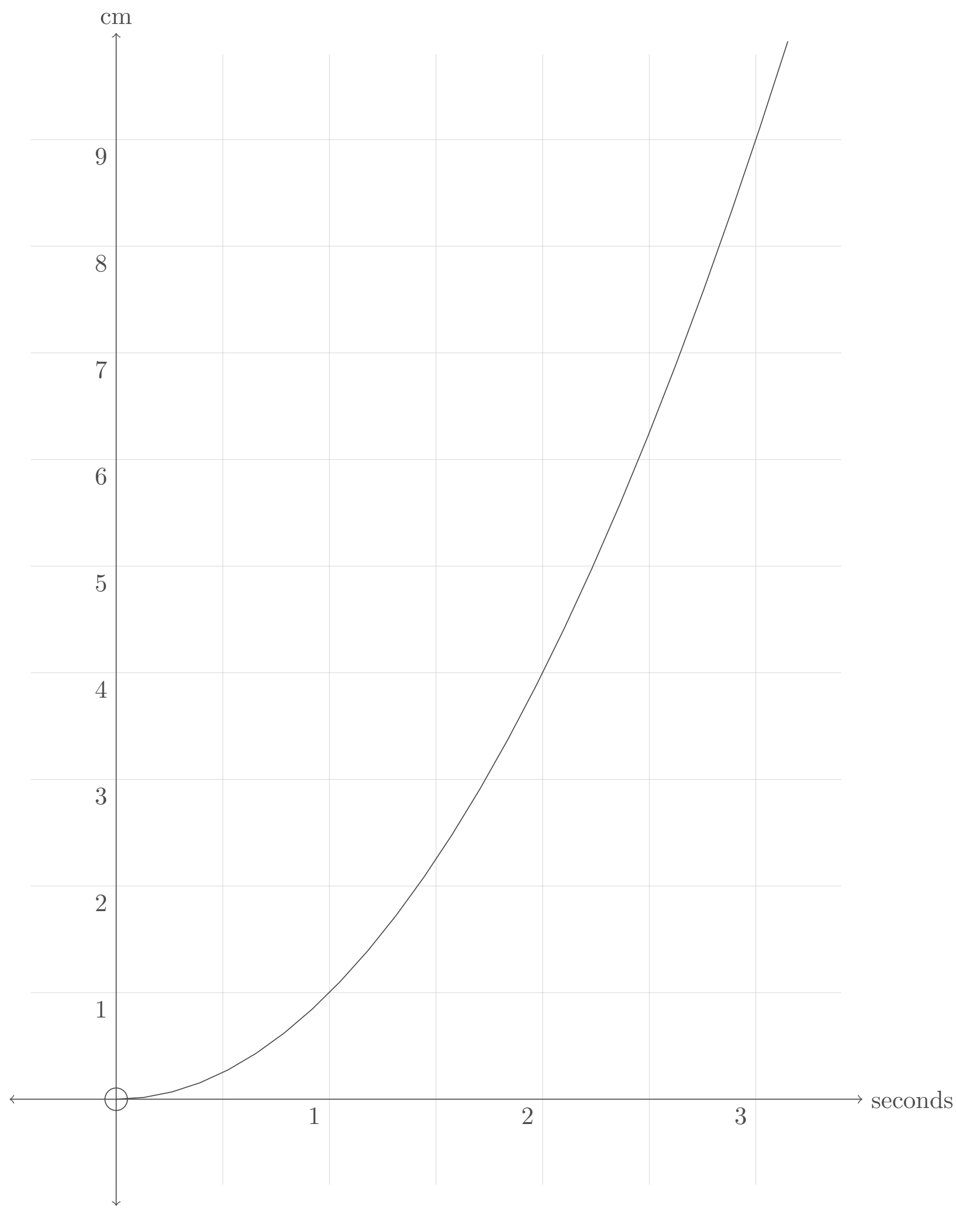
Instantaneous rates of change

Average rates of change are very easy to find but only so useful, as they only give us a rough idea of how something is changing.

What we really want to know is the instantaneous rate of change at a given point in time. The rate of change at an instant is difficult to find because an instant lasts for 0 seconds! No time passes in an instant. This makes our lives very difficult because we define a rate of change as the amount by which something changes over a certain amount of time.

If there is no amount of time in an instant then how can we talk about a rate of change at an instant? First I will show you that a rate of change can exist at an instant and then I will show you how to find it.

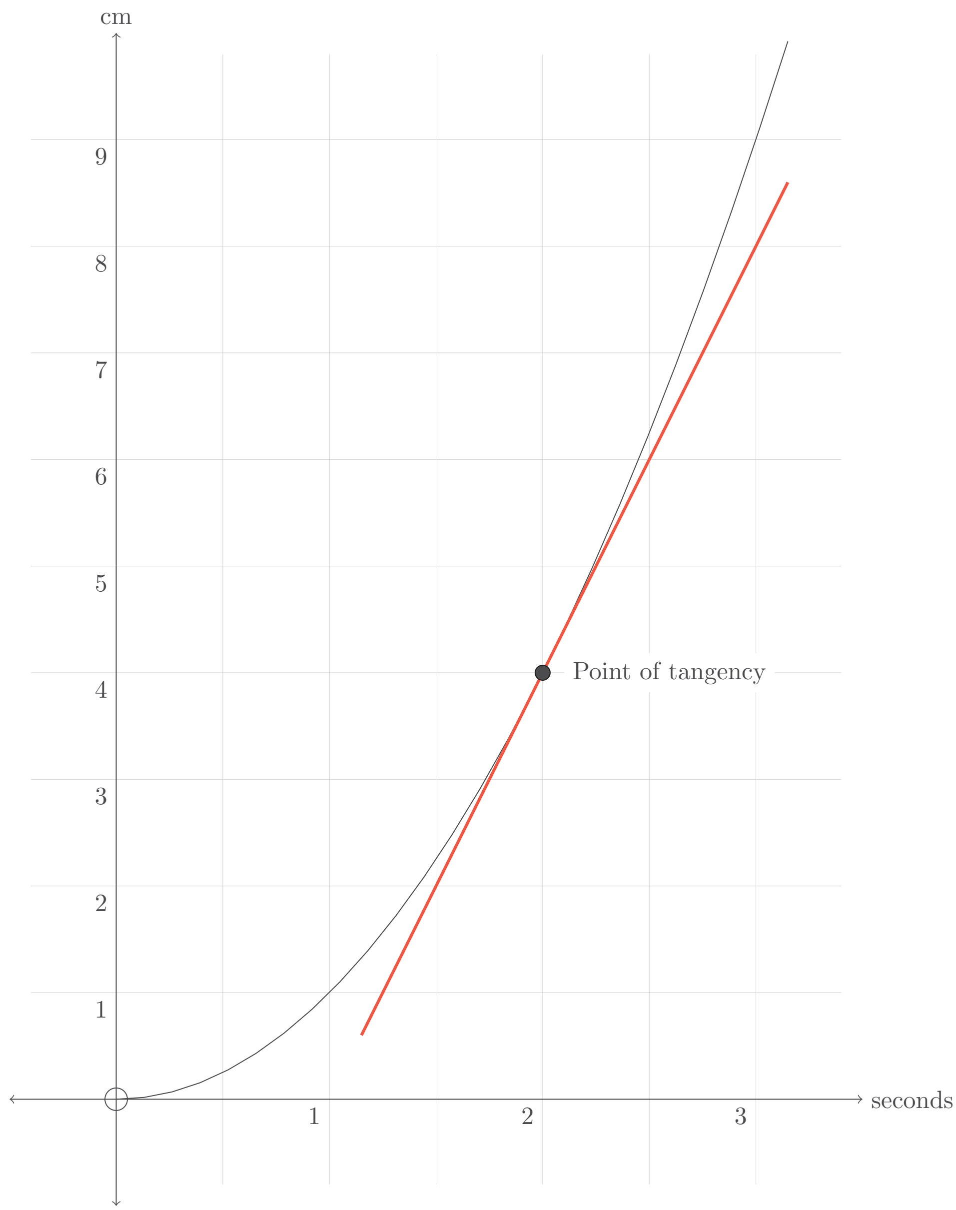
Opposite is the graph of the position of our mouse over time, when his rate of change of position is changing (see question 8).



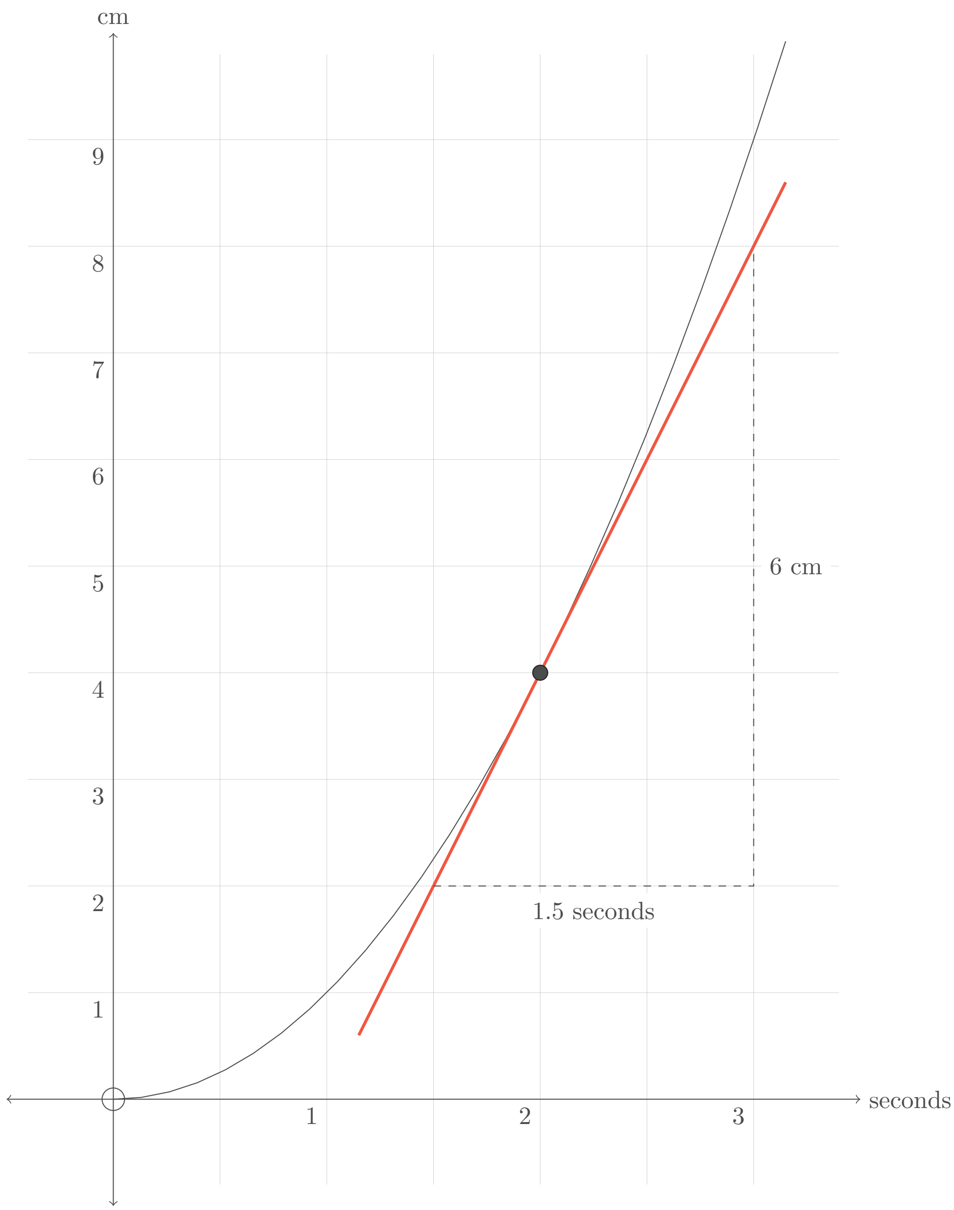
Let's ask ourselves, "What is his instantaneous rate of change at 2 seconds?"

So far we have seen that a rate of change can always be represented by the gradient of a straight line. So what we should be looking for is a straight line. But where should it be?

The straight line that represents the instantaneous rate of change at 2 seconds is the tangent to the curve at 2 seconds:



If we find the gradient of this line then we will have found the instantaneous rate of change of the mouse's position at time = 2 seconds.

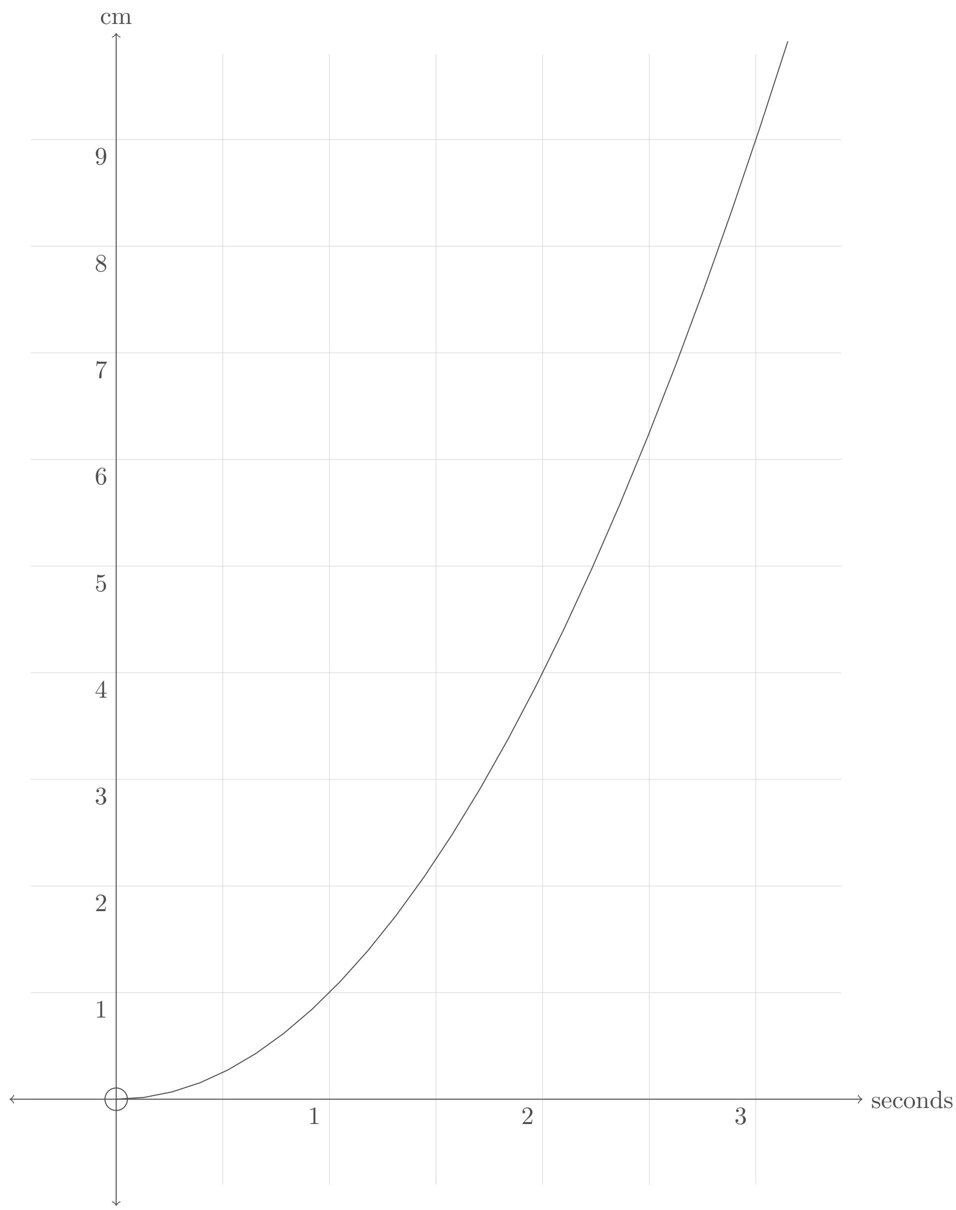


$$\text{Gradient of tangent} = \frac{6\ cm}{1.5\ sec}$$
$$= 4\ cm/sec$$

Therefore, at 2 seconds, the instantaneous rate of change of the mouse's position is 4 cm/sec.

Question 12:

Use this method and the graph that follows to find the instantaneous rate of change of position of the mouse at 1 second: (Use a ruler to draw the tangent as accurately as you can.)



This method is not always that easy to use. Is there not a more mathematical way of finding the instantaneous rate of change of a graph? We could try to use the formula:

$$\text{Rate of change} = \frac{\Delta\text{Postion}}{\Delta\text{Time}}$$

But if we are talking about an instant in time then this is what we get if we take our instantaneous point to be (1,1):

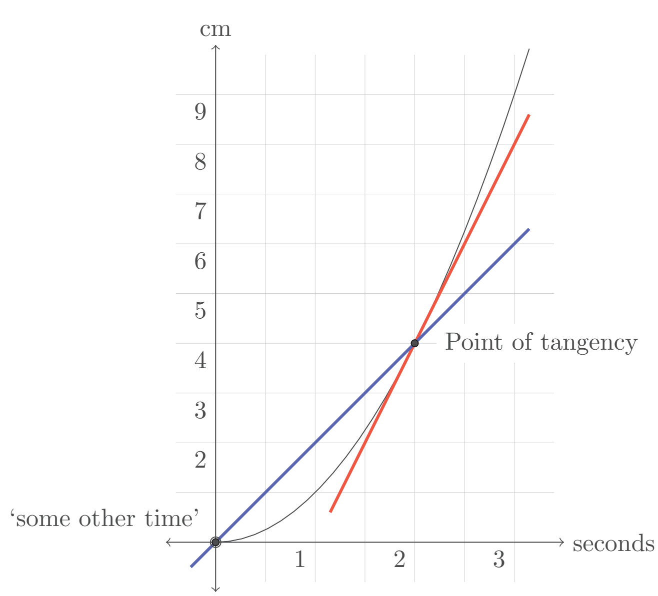
$$\begin{aligned} &\frac{\Delta\text{Postion}}{\Delta\text{Time}} \\ &= \frac{1-1}{1-1} \\ &= \frac{0}{0} \end{aligned}$$

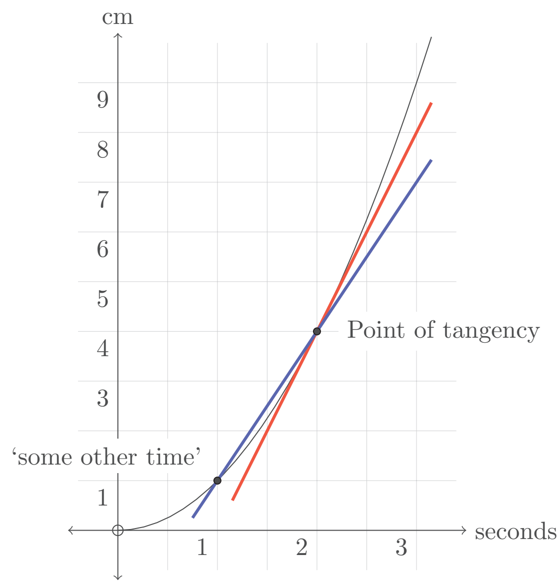
That doesn't work! So we can't use the formula to help us. However, what did we do in the past when we came across $\frac{0}{0}$? Limits should come to mind.

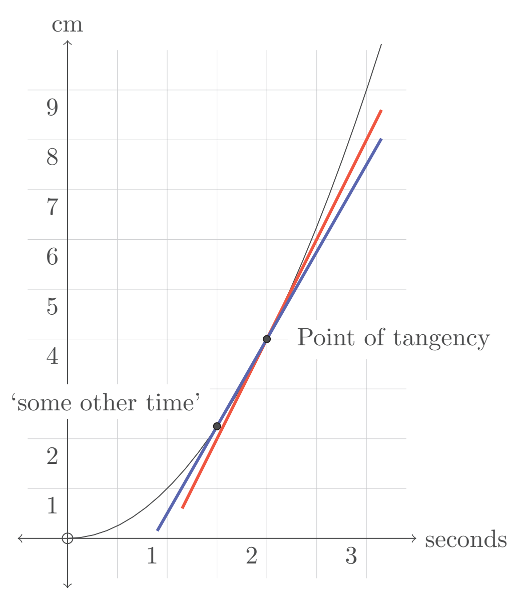
A crafty method

We will now look at a few graphs that will start to give you an idea of how we can work out an instantaneous rate of change without having to draw a tangent at all. We will still use tangents and straight lines to explain the idea but once you have the idea you will see that all we actually need is a formula that uses limits. With that formula we will be able to calculate the gradient of any tangent, without having to draw anything. Right now, however, I just want you to grasp the idea and then we will look at the formula in the next lesson.

In each graph below we see the red tangent to the graph at time = 2 seconds, which represents the mouse's instantaneous rate of change of position at 2 seconds. We also see the blue line representing the mouse's average rate of change of position between 2 seconds and 'some other time' on the graph. As you look through graphs, you will see that the 'other time' that the blue line passes through, moves closer and closer to 2 seconds so that the blue line, representing the mouse's average rate of change of position in each graph, looks more and more like the red tangent.




cm
9
8
7
6
5
4
3
1
'some other time'
Point of tangency
1
2
3
seconds


cm
9
8
7
6
5
4
2
1
'some other time'
Point of tangency
1
2
3
seconds

Question 13:

Calculate the gradients of the blue lines representing the mouse's average rate of change of position (average velocity) in each graph:

	Gradient of blue line
First graph	
Second graph	
Third graph	

Question 14:

Compare these average rates of change to the instantaneous rate of change at time = 2 seconds ($4cm/sec$). What do you notice?

Question 15:

What is the limit of the blue line representing the average rate of change, as the point at 'some other time' approaches the point of tangency?

In the next lesson we will use this idea to determine a formula which will allow us to find the exact gradient of any tangent to a function without having to use the method of drawing the tangent line.

Solutions

Solution 1:

$$\frac{\Delta\text{Volume}}{\Delta\text{Time}} = \frac{75-50}{150} = \frac{25}{150} = \frac{1}{6}ml/sec$$

Solution 2:

$$\frac{\Delta\text{Volume}}{\Delta\text{Time}} = \frac{1400-200}{3} = \frac{1200}{3} = 400cm^3/min$$

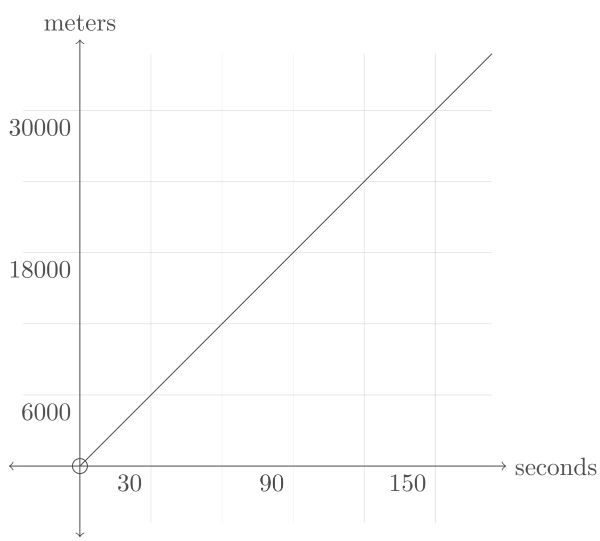
Solution 3:

$$\frac{\Delta\text{Volume}}{\Delta\text{Time}} = \frac{6-1}{5} = \frac{5}{5} = 1l/sec$$

Solution 4:

$$\frac{\Delta\text{Distance}}{\Delta\text{Time}} = \frac{30km}{13:02:30-13:00:00} = \frac{30000}{150} = 200m/sec$$

Solution 5:



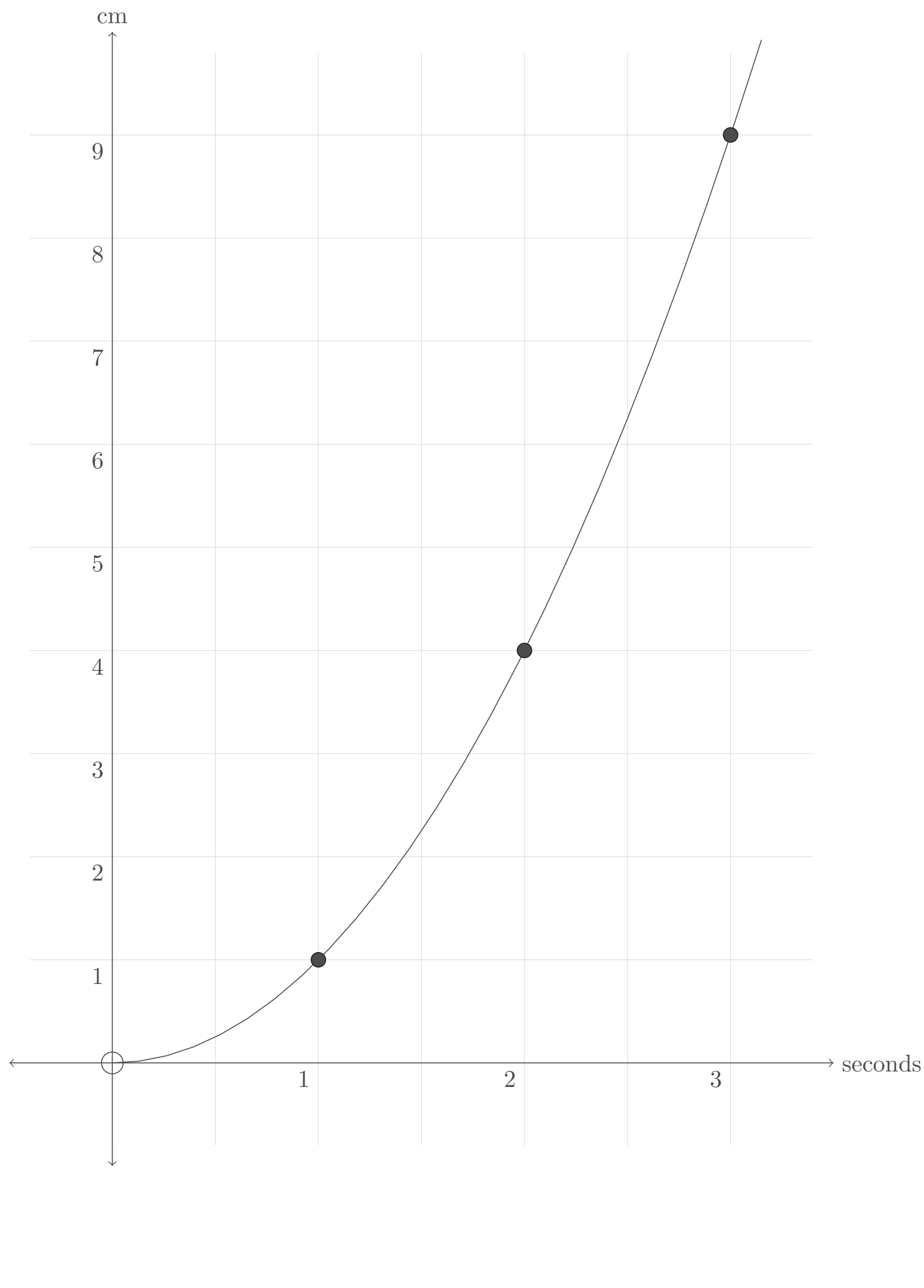
Solution 6:

Rate of change	Gradient
$200m/sec$	$\frac{y_1-y_2}{x_1-x_2}$ $=\frac{18000-6000}{90-30}$ $=\frac{12000}{60}$ $=200m/sec$

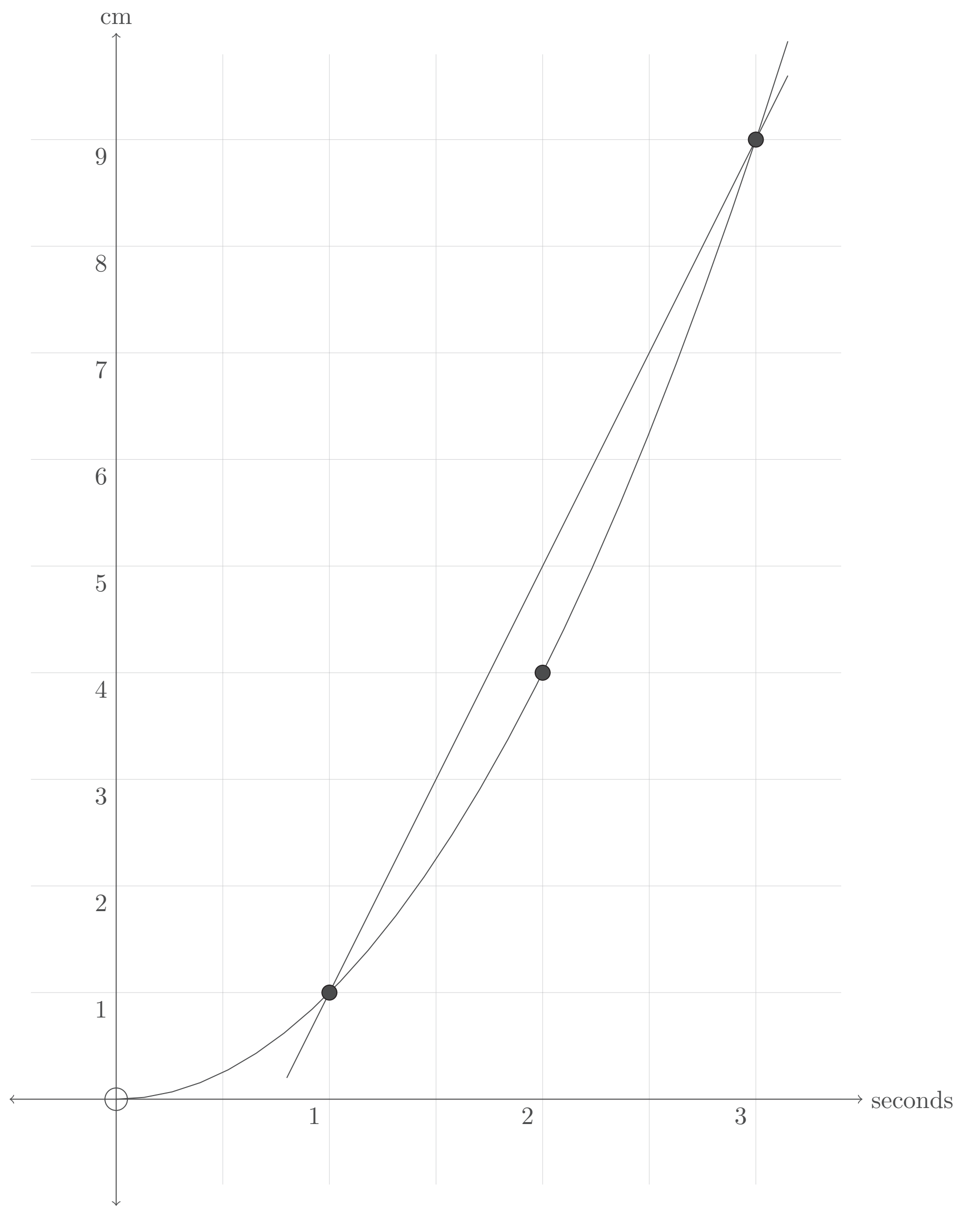
Solution 7:

$$\frac{\Delta\text{Position}}{\Delta\text{Time}} = \frac{4}{2} = 2cm/sec$$

Solution 8:



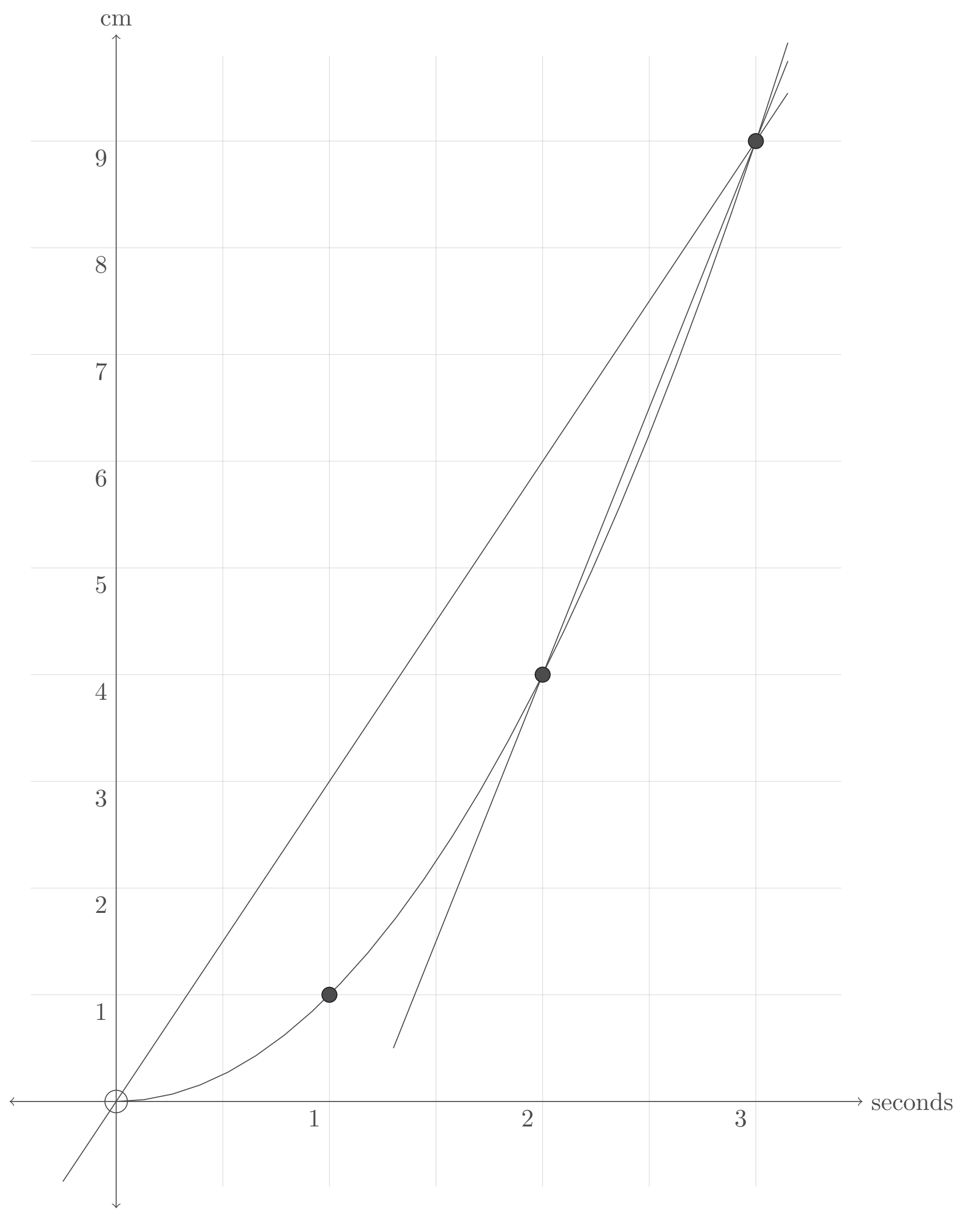
Solution 9:



$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{9-1}{3-1} = \frac{8}{2} = 4m/sec$$

Solution 10:

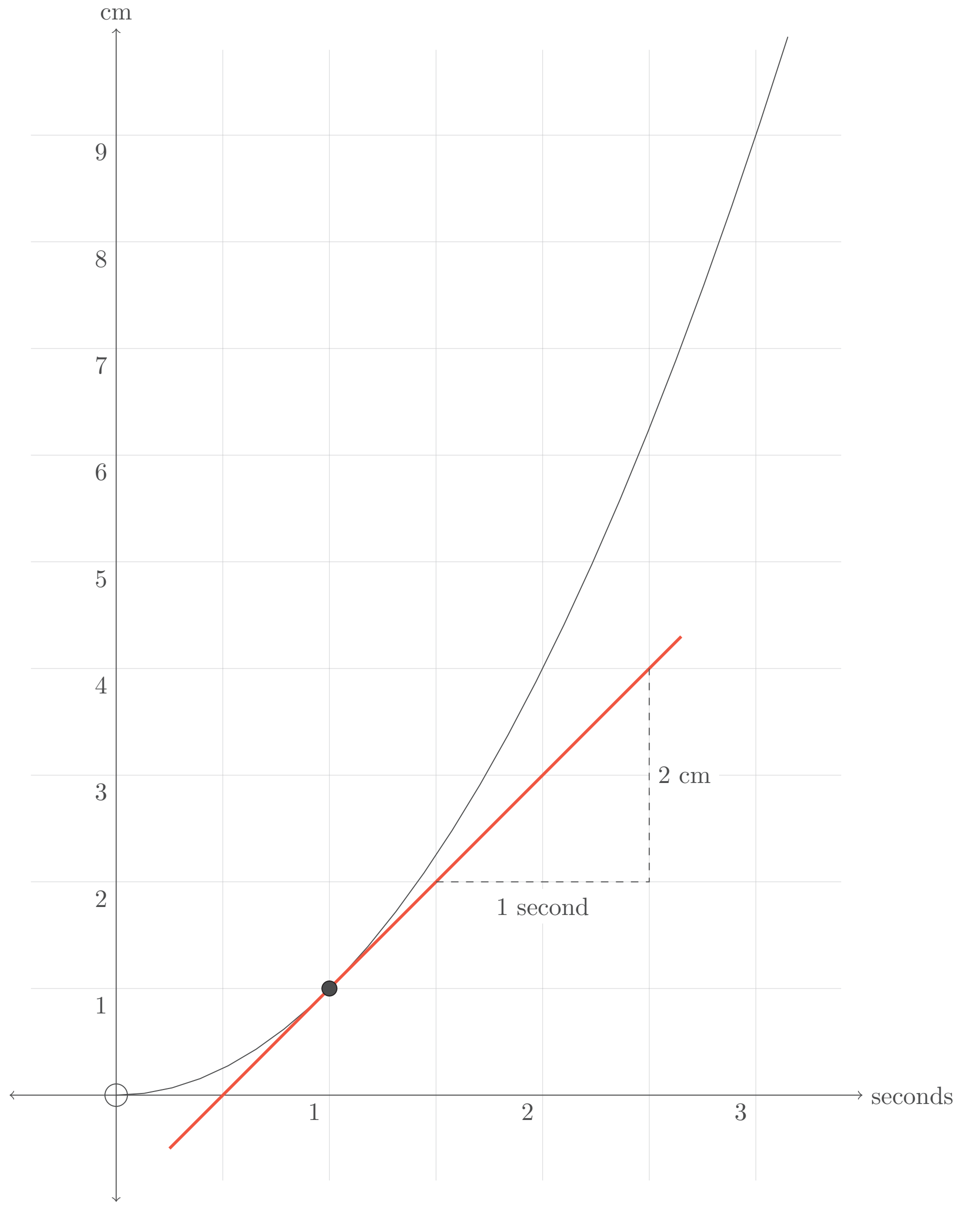
Time interval	Average rate of change
0 to 3 seconds	$\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{9-0}{3-0} = \dfrac{9}{3} = 3m/sec$
2 to 3 seconds	$\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{9-4}{3-2} = \dfrac{5}{1} = 5m/sec$



Solution 11:

Time interval	Average rate of change
0 to 4 seconds	$\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{10-0}{4-0} = \dfrac{10}{4} = 2.5cm^2/sec$
0 to 1 second	$\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{4-0}{1-0} = \dfrac{4}{1} = 4cm^2/sec$
1 to 3 seconds	$\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{9-4}{3-1} = \dfrac{5}{2} = 2.5cm^2/sec$
2 to 4 seconds	$\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{10-7}{4-2} = \dfrac{3}{2} = 1.5cm^2/sec$

Solution 12:



$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{4-2}{2.5-1.5} = \frac{2}{1} = 2cm/sec$$

Solution 13:

	Gradient of blue line
First graph	$\frac{y_1 - y_2}{x_1 - x_2} = \frac{4-0}{2-0} = \frac{4}{2} = 2cm/sec$
Second graph	$\frac{y_1 - y_2}{x_1 - x_2} = \frac{4-1}{2-1} = \frac{3}{1} = 3cm/sec$
Third graph	$\frac{y_1 - y_2}{x_1 - x_2} = \frac{4-2.25}{2-1.5} = \frac{1.75}{0.5} = 3.5cm/sec$

Solution 14:

The gradient of the blue line is getting closer to the gradient of the red tangent.

Solution 15:

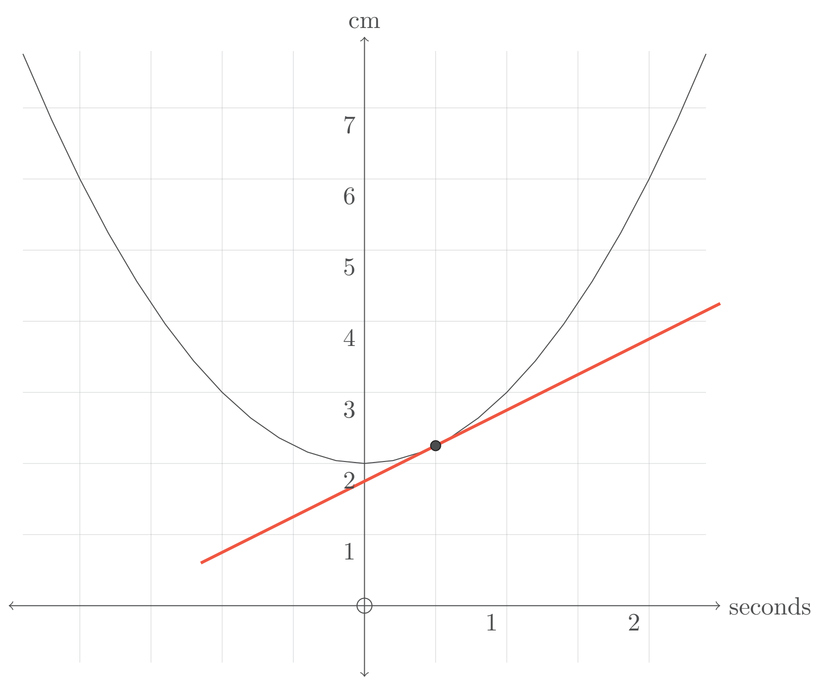
The limit of the blue line, as the point at 'some other time' approaches the point of tangency, is the red tangent.

Differentiation

Lesson 2: First principles and the polynomial rule

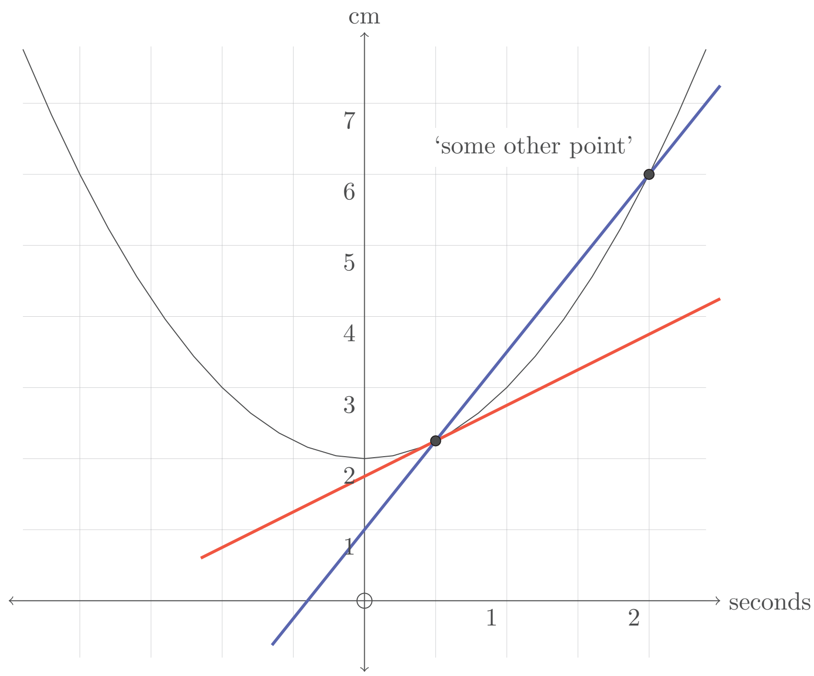
First principles is the method we use to find an instantaneous rate of change using limits. The graph of $f(x)$ below will be used as an example and we will determine the instantaneous rate of change at 0.5 seconds (shown by the red tangent).

$$f(x) = x^2 + 2$$



We start out by drawing a blue line. We let the blue line pass through the same point that the red tangent touches and some other point.

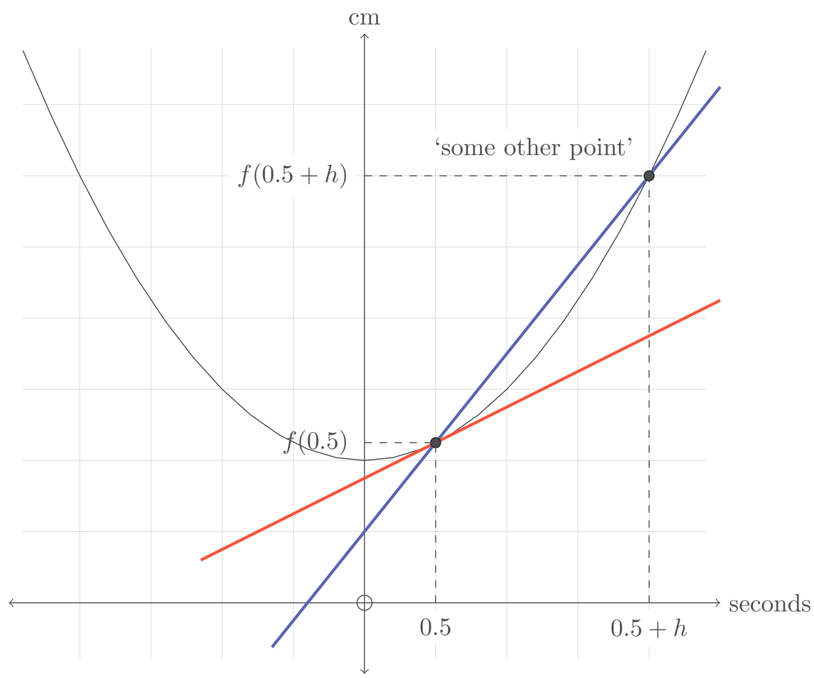
$$f(x) = x^2 + 2$$



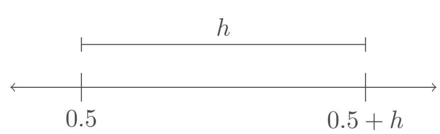
The useful thing about the blue line is that, if we have the value of the other point that it passes through (labeled 'some other point' on the graph), we can use the gradient formula, $\frac{y_1 - y_2}{x_1 - x_2}$ to find its gradient.

I'm now going to give you the value of the other point, but it may look a bit strange:

$f(x) = x^2 + 2$



What is h?... h is a variable, h is the distance from 0.5 to 0.5 + h.



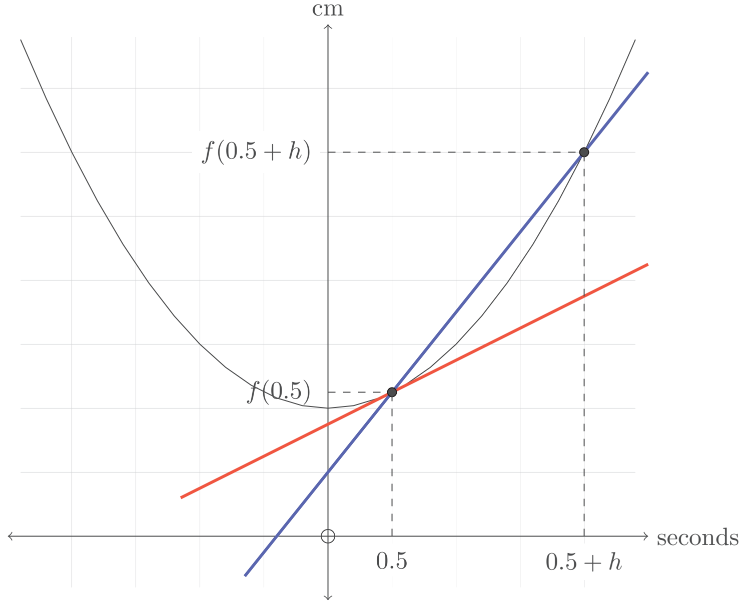
But why did I write the x value of the other point in such a strange way? Follow the reasoning below very carefully and it will become clear. This reasoning forms the foundation of how differentiation by first principles works.

The core idea

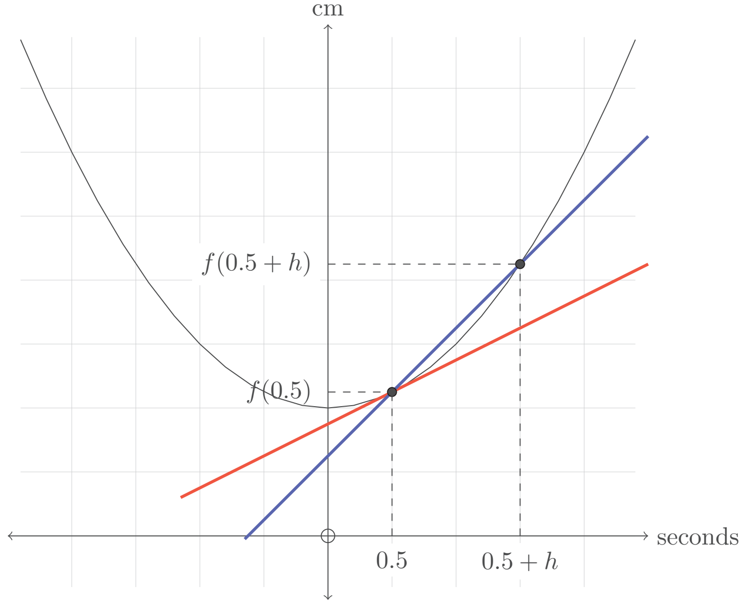
What happens as h gets smaller and smaller?

The smaller h gets, the more the blue line looks like the red tangent. And the more the blue line looks like the red tangent, the closer the value of the gradient of the blue line is to the value of the gradient of the red tangent. The following three graphs illustrate this process.

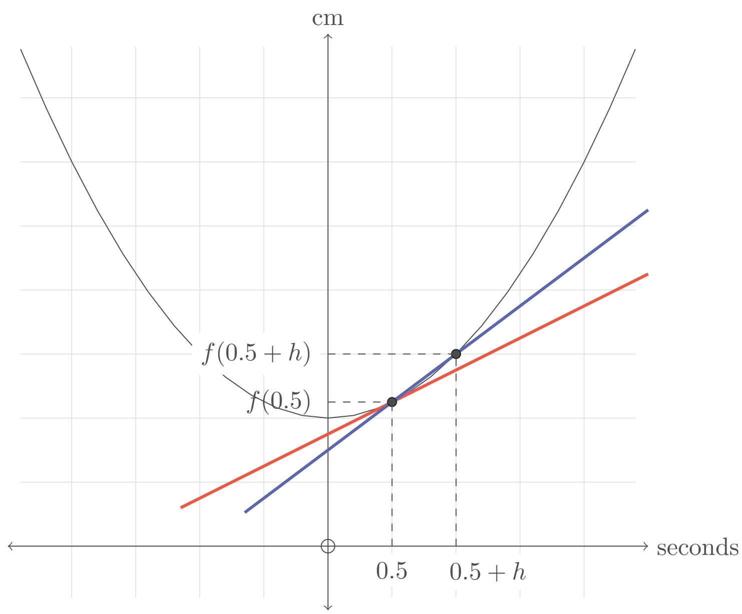
$h = 1.5$



$h = 1$



$h = 0.5$



If we write out the gradient of the blue line in terms of h and then find the limit of the gradient of the blue line, as h approaches 0, that limit will be equal to the gradient of the red tangent.

*** WARNING: Do not read further until you have grasped this core idea! ***

Writing this all down in a formula, we get:

$$\boxed{\text{Gradient of red tangent} = \lim_{h \to 0} \frac{f(0.5+h) - f(0.5)}{(0.5+h) - 0.5}}$$

Here is a more detailed explanation of how we arrived at this formula:

We know that $m = \dfrac{y_1 - y_2}{x_1 - x_2}$

We have two points on our blue line:

$$(x_1, y_1) = (0.5 + h, f(0.5 + h))$$
$$(x_2, y_2) = (0.5, f(0.5))$$

So we can express the gradient of the blue line as:

$$m = \frac{f(0.5+h) - f(0.5)}{(0.5+h) - 0.5}$$

The smaller h becomes, the closer this gradient gets to the gradient of the red tangent. Therefore, if we make h really small then the value of will be really close to the value of the gradient of the red tangent and that's why:

$$\text{Gradient of red tangent} = \lim_{h \to 0} \frac{f(0.5+h) - f(0.5)}{(0.5+h) - 0.5}$$

Finding the limit

All that is left is to work out what $\lim_{h \to 0} \frac{f(0.5+h)-f(0.5)}{(0.5+h)-0.5}$ is actually equal to, remembering that $f(x) = x^2 + 2$.

First we simplify:

$$\begin{aligned} &\lim_{h \to 0} \frac{f(0.5+h)-f(0.5)}{(0.5+h)-0.5} \\ =&\lim_{h \to 0} \frac{[(0.5+h)^2+2]-[(0.5)+2]}{0.5+h-0.5} \\ =&\lim_{h \to 0} \frac{0.25+h+h^2+2-0.25-2}{h} \\ =&\lim_{h \to 0} \frac{h+h^2}{h} \end{aligned}$$

Then we perform a bit of surgery:

$$\begin{aligned} =&\lim_{h \to 0} \frac{h(1+h)}{h} \\ =&\lim_{h \to 0}(1+h) \end{aligned}$$

Once the surgery is done we can substitute in 0:

$$\begin{aligned} =&1+0 \\ =&1 \end{aligned}$$

Therefore the gradient of the red tangent is equal to 1! Which can be interpreted as: the instantaneous rate of change of position at 0.5 seconds is $1cm/sec$. Unlike the graphical method of finding the gradient, differentiation by first principles will always give us an exact value.

If you have understood everything up until here then you have understood differentiation by first principles.

Terminology and notation

If we use limits to find the instantaneous rate of change of function, it is called differentiation and $\lim_{h \to 0} \frac{f(0.5+h)-f(0.5)}{(0.5+h)-0.5}$ is called **Newton's quotient**. If you are asked to differentiate $f(x)$ at $x = 0.5$, it could be written like this:

$$f'(0.5)$$

Or like this:

$$\frac{d}{dx}f(0.5)$$

$f'(0.5)$ and $\frac{d}{dx}f(0.5)$ both mean, 'differentiate $f(x)$ at $x = 0.5$'.

Question 1:

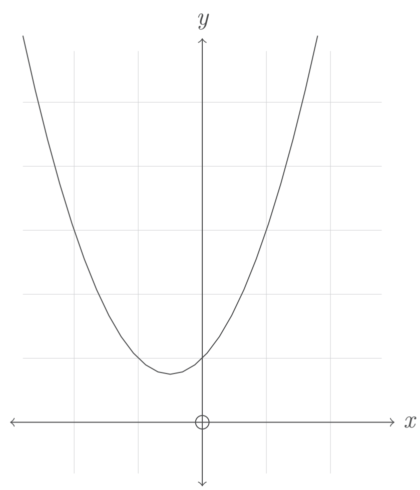
Use first principles to differentiate each function by finding Newton's quotient, solving it and then write the gradient in the table:

	Newton's quotient: $\left(\lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{(c+h)-c}\right)$	Gradient
Differentiate $f(x) = -x^2 + 4$ at x=3		
Differentiate $f(x) = x^2 - 2x + 1$ at x=-1		
Find $f'(5)$ for $f(x) = 3x^2 - 2$		
Find $f'(0)$ for $f(x) = x^2$		

Question 2:

Now that we are comfortable with differentiating a function at a given point, use first priciples to differentiate $f(x) = x^2 + x + 1$ at: $x = -2$, $x = 0$ and $x = 3$. Once again, find Newton's quotient, solve it and then write the gradient in the table:

$$f(x) = x^2 + x + 1$$



	Newton's quotient: $\left(\lim_{h \to 0} \frac{f(c+h) - f(c)}{(c+h) - c}\right)$	Gradient
At $x = -2$		
At $x = 0$		
At $x = 3$		

The derivative of a function

In question 2, each time you differentiated $f(x) = x^2 + x + 1$ at a different value for x, you had to go through the whole method of first principles.

However, there is a shortcut. It is possible to find a 'differentiation formula' for a function. A 'differentiation formula' would mean that, every time you wanted to find the gradient of $f(x) = x^2 + x + 1$ at a specific value of x, all you would need to do is plug that value of x into the 'differentiation formula' and you would instantly get the value of the gradient of $f(x) = x^2 + x + 1$ at that value of x, without having to go through the method of first priciples each time.

But how do we find such a 'differentiation formula'? We use the exact same method as before! The only difference is that, instead of choosing a specific value for x, we just leave x as x.

Before, if we wanted to differentiation $f(x) = x^2 + x + 1$ at $x = 3$, we wrote:

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{(3+h) - 3}$$

Now, if we want to find the 'differentiation formula' of $f(x) = x^2 + x + 1$, we write:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

Can you see that, instead of using a specific value, we have just used x. Now all we do is solve Newton's quotient using the same method as before:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} \\
&= \lim_{h \to 0} \frac{[(x+h)^2 + (x+h) + 1] - [x^2 + x + 1]}{x + h - x} \\
&= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 + x + h + 1 - x^2 - x - 1}{h} \\
&= \lim_{h \to 0} \frac{2hx + h^2 + h}{h} \\
&= \lim_{h \to 0} \frac{h(2x + h + 1)}{h} \\
&= \lim_{h \to 0} (2x + h + 1) \\
&= 2x + 1
\end{aligned}$$

$f'(x) = 2x + 1$ is the 'differentiation formula' for $f(x) = x^2 + x + 1$. However, we don't usually call it a 'differentiation formula'. We call it a **derivative**.

$f'(x) = 2x + 1$ is the derivative $f(x) = x^2 + x + 1$.

Question 3:

Use $f'(x) = 2x + 1$ to find the gradient of $f(x) = x^2 + x + 1$ at the following points:

	Gradient
At $x = -2$	
At $x = 0$	
At $x = 3$	

Now compare the answers that you get in question 3 with the values of the gradients that you worked out in question 2. It is much easier to find the derivative of a function and use it to find the gradient of the function at different points as opposed to differentiating a function at each point individually. In fact, even if you only had to differentiate a function at one point, you might as well find the derivative for the function and then plug the x value for that point into the derivative.

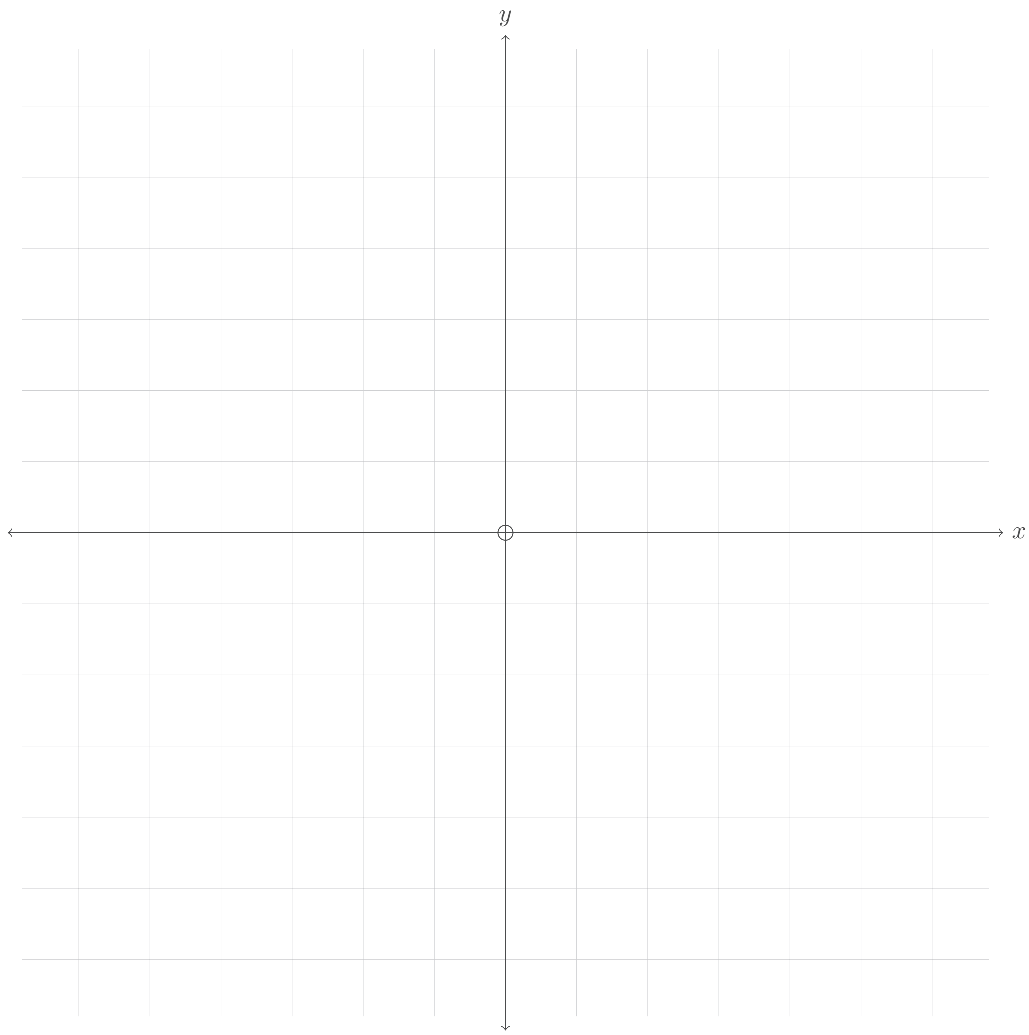
Question 4:

Use first priciples to find the derivatives of the following functions:

x^2	
$x^2 - 5x + 5$	
$-3x^2 - 1$	

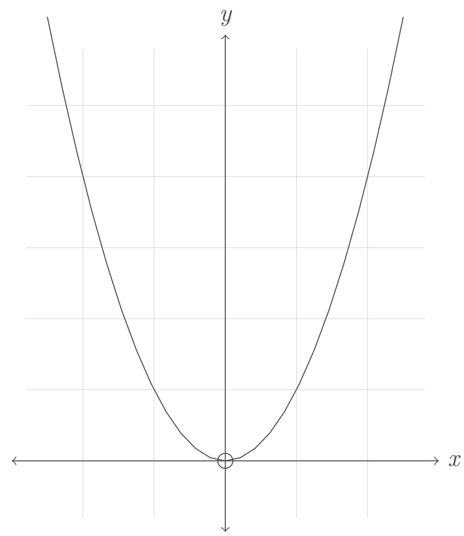
Question 5:

Plot the derivatives that you found in question 4 on the set of axes below:



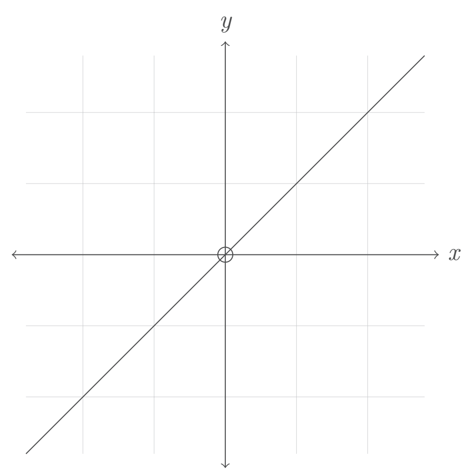
It can be seen from question 5 that a derivative is itself a function. Let's look at this idea in more detail:

Here is the graph of the function, $f(x) = x^2$:



You have already worked out its derivative, $f'(x) = 2x$.

This is the graph of $f'(x) = 2x$:



The value of $f'(x)$ at $x = 1$ is 2. The gradient of $f(x)$ at $x = 1$ is 2.
The value of $f'(x)$ at $x = 2$ is 4. The gradient of $f(x)$ at $x = 2$ is 4.
The value of $f'(x)$ at $x = 3$ is 6. The gradient of $f(x)$ at $x = 3$ is 6.
And so on...

In general, the function value of $f'(x)$ at any x value is the gradient value of $f(x)$ for the same x value. In this example, if you wanted the gradient value of the parabola at $x = 1$, you would look at the value of the straight line graph at $x = 1$, which is 2, and that tells you that the gradient of the parabola, at $x = 1$, is equal to 2.

The polynomial rule

There is an easier method to find the derivative of a polynomial. This new method only works on polynomials and that is why it is called the polynomial rule.

The polynomial rule:

For each term of the polynomial we multiply the co-efficient by the exponent and subtract 1 from the exponent. If the term is a constant we make it a 0.

Example:

$$\begin{aligned} f(x) &= 3x^2 + 2x + 1 \\ \therefore f'(x) &= 6x + 2 \end{aligned}$$

Let's go through that a bit slower:

$$\begin{aligned} f'(x) &= (2 \times 3)x^{2-1} + (1 \times 2)x^{1-1} + 0 \\ &= 6x^1 + 2x^0 \\ &= 6x + (2 \times 1) \\ &= 6x + 2 \end{aligned}$$

Now wasn't that a lot quicker than first priciples! You may be wondering why we bother even learning first principles when the polynomial rule is so much easier to use. First priciples teaches us an understanding of how differentiation works whereas the polynomial rule does not. The polynomial rule makes our work a lot easier but first principles gives us a deeper understanding of the work, which builds our confidence in the theory. That being said, we will spend the last part of the lesson using the polynomial rule.

Question 6:

To see how useful this method is, try it out on these three functions and compare your answers to those that you got using first principles in question 4:

x^2	
$x^2 - 5x + 5$	
$-3x^2 - 1$	

The polynomial rule can be used to differentiate any kind of polynomial.

Question 7:

Use the polynomial rule to differentiate the following polynomials:

$x^2 + 1$	
$x + 5$	
$-5x^2 + 1$	
$x^5 - x^3 + 1$	
$5x^{101} - x^{50} + 3$	
$x^3 + x^2 + x + 1$	
$x^3 - x$	
$-3x - 2$	
2	
-10	

It is also possible to differentiate many other types of functions: exponentials, hyperbolas, rational, trigonometric and many more. We have only looked at polynomials in this module but in your first year mathematics course you will learn about other rules used to differentiate other types of functions. Differentiation can also be used to solve many types of real world problems. It is a very big section in first year as well as being very exciting!

Solutions

Solution 1:

	Newton's quotient: $\left(\lim\limits_{h\to 0}\dfrac{f(c+h)-f(c)}{(c+h)-c}\right)$	Gradient
Differentiate $f(x) = -x^2+4$ at $x=3$	$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{[-(3+h)^2+4]-[-(3)^2+4]}{(3+h)-3} \\ &= \lim_{h\to 0}\frac{-3^2-6h-h^2+4+3^2-4}{h} \\ &= \lim_{h\to 0}\frac{-6h-h^2}{h} \\ &= \lim_{h\to 0}\frac{h(-6-h)}{h} \\ &= \lim_{h\to 0}(-6-h) \\ &= -6 \end{aligned}$	-6
Differentiate $f(x) = x^2-2x+1$ at $x=-1$	$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{[(-1+h)^2-2(-1+h)+1]-[(-1)^2-2(-1)+1]}{(-1+h)-(-1)} \\ &= \lim_{h\to 0}\frac{1-2h+h^2+2-2h+1-1-2-1}{h} \\ &= \lim_{h\to 0}\frac{-4h+h^2}{h} \\ &= \lim_{h\to 0}\frac{h(-4+h)}{h} \\ &= \lim_{h\to 0}(-4+h) \\ &= -4 \end{aligned}$	-4
Find $f'(5)$ for $f(x) = 3x^2-2$	$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{[3(5+h)^2-2]-[3(5)^2-2]}{(5+h)-5} \\ &= \lim_{h\to 0}\frac{75+30h+3h^2-2-75+2}{h} \\ &= \lim_{h\to 0}\frac{30h+3h^2}{h} \\ &= \lim_{h\to 0}\frac{h(30+3h)}{h} \\ &= \lim_{h\to 0}(30+3h) \\ &=30 \end{aligned}$	30
Find $f'(0)$ for $f(x) = x^2$	$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{[(0+h)^2]-[(0)^2]}{(0+h)-0} \\ &= \lim_{h\to 0}\frac{h^2}{h} \\ &= \lim_{h\to 0}h \\ &=0 \end{aligned}$	0

Solution 2:

	Newton's quotient: $\left(\lim\limits_{h\to 0}\dfrac{f(c+h)-f(c)}{(c+h)-c}\right)$	Gradient
At $x=-2$	$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{[(-2+h)^2+(-2+h)+1]-[(-2)^2+(-2)+1]}{(-2+h)-(-2)} \\ &= \lim_{h\to 0}\frac{4-4h+h^2-2+h+1-4+2-1}{h} \\ &= \lim_{h\to 0}\frac{-3h+h^2}{h} \\ &= \lim_{h\to 0}\frac{h(-3+h)}{h} \\ &= \lim_{h\to 0}(-3+h) \\ &= -3 \end{aligned}$	-3
At $x=0$	$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{[(0+h)^2+(0+h)+1]-[(0)^2+(0)+1]}{(0+h)-0} \\ &= \lim_{h\to 0}\frac{h^2+h+1-1}{h} \\ &= \lim_{h\to 0}\frac{h+h^2}{h} \\ &= \lim_{h\to 0}\frac{h(1+h)}{h} \\ &= \lim_{h\to 0}(1+h) \\ &=1 \end{aligned}$	1
At $x=3$	$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{[(3+h)^2+(3+h)+1]-[(3)^2+(3)+1]}{(3+h)-3} \\ &= \lim_{h\to 0}\frac{9+6h+h^2+3+h+1-9-3-1}{h} \\ &= \lim_{h\to 0}\frac{7h+h^2}{h} \\ &= \lim_{h\to 0}\frac{h(7+h)}{h} \\ &= \lim_{h\to 0}(7+h) \\ &=7 \end{aligned}$	7

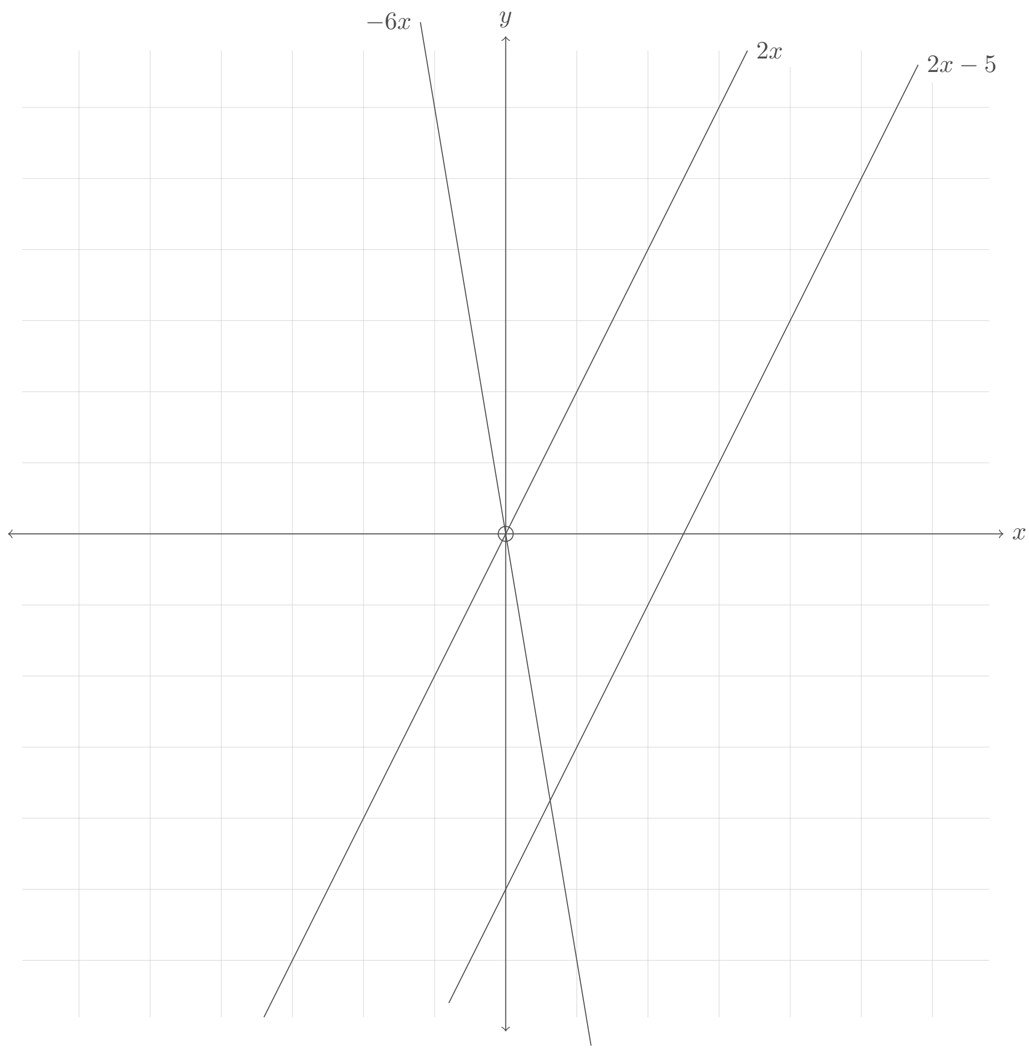
Solution 3:

	Gradient
At $x = -2$	-3
At $x = 0$	1
At $x = 3$	7

Solution 4:

x^2	$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[(x+h)^2] - [x^2]}{(x+h) - x} \\ &= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0}(2x+h) \\ &= 2x \end{aligned}$
$x^2 - 5x + 5$	$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[(x+h)^2 - 5(x+h) + 5] - [x^2 - 5x + 5]}{(x+h) - x} \\ &= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - 5x - 5h + 5 - x^2 + 5x - 5}{h} \\ &= \lim_{h\to 0} \frac{2hx + h^2 - 5h}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h-5)}{h} \\ &= \lim_{h\to 0}(2x+h-5) \\ &= 2x - 5 \end{aligned}$
$-3x^2 - 1$	$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[-3(x+h)^2 - 1] - [-3x^2 - 1]}{(x+h) - x} \\ &= \lim_{h\to 0} \frac{-3x^2 - 6hx - 3h^2 - 1 + 3x^2 + 1}{h} \\ &= \lim_{h\to 0} \frac{-6hx - 3h^2}{h} \\ &= \lim_{h\to 0} \frac{h(-6x-3h)}{h} \\ &= \lim_{h\to 0}(-6x-3h) \\ &= -6x \end{aligned}$

Solution 5:



Solution 6:

x^2	$2x$
x^2-5x+5	$2x-5$
$-3x^2-1$	$-6x$

Solution 7:

x^2+1	$2x$
$x+5$	1
$-5x^2+1$	$-10x$
x^5-x^3+1	$5x^4-3x^2$
$5x^{101}-x^{50}+3$	$505x^{100}-50x^{49}$
x^3+x^2+x+1	$3x^2+2x+1$
x^3-x	$3x^2-1$
$-3x-2$	-3
2	0
-10	0

Integration

Lesson 1: First principles

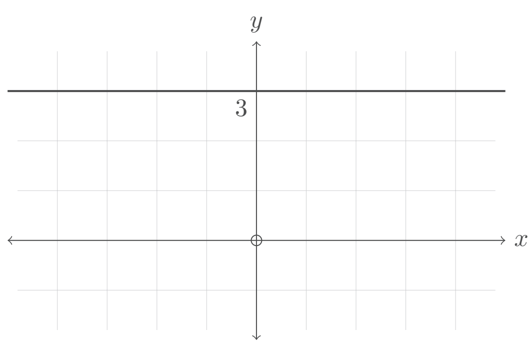
We have arrived at the last section and what a section it will be. The idea behind integration is very simple; the mathematics is a bit trickier.

The idea

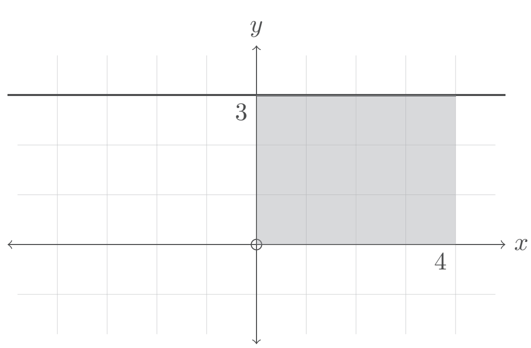
We want to be able to find the area under any curve (between the curve itself and the x-axis) and between two values on the x-axis. The following questions will make this idea clear.

Question 1:

I give you the graph of $y = 3$:



and then I ask you to calculate the area under the curve, between $x = 0$ and $x = 4$ (the shaded area):

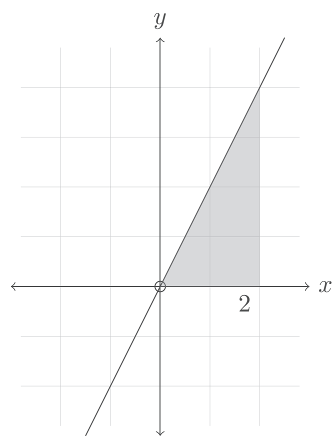


Area:

(Don't worry about the units of the area. Usually they would depend on the units of the x and y axes, but right now we want to focus on the more important details.)

Question 2:

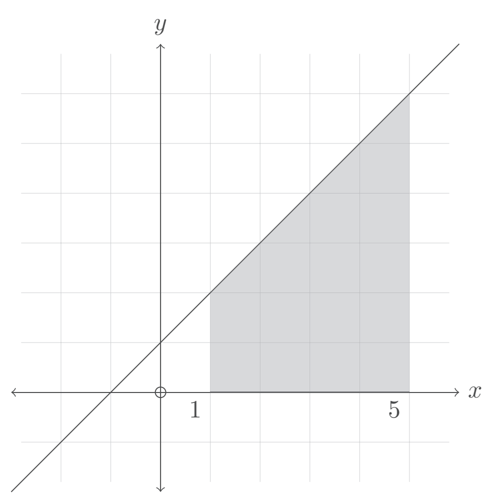
Let's try that again but this time I will give you the graph of $y = 2x$ and I'll ask you to calculate the area under the curve, between $x = 0$ and $x = 2$.



Area:

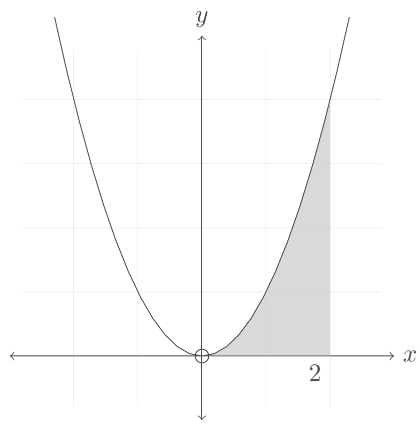
Question 3:

Let's try an even trickier one. I give you the graph of $y = x + 1$ and I ask you to calculate the area under the curve, between $x = 1$ and $x = 5$.



Area:

Let's try an EVEN trickier one! I give you the graph of $y = x^2$ and I ask you to calculate the area under the curve, between $x = 0$ and $x = 2$. Can you do it?

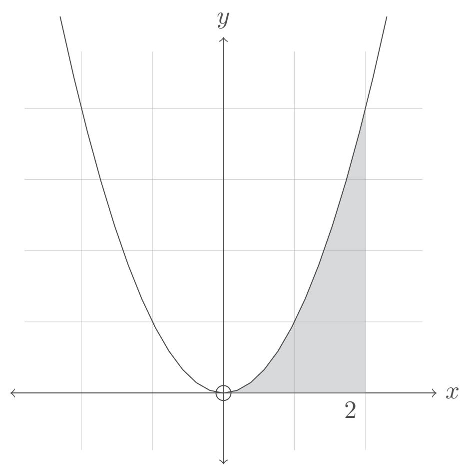


Not yet! Finding the area under the first three curves is easy because they are all staight lines, so all we have to do is work out the area of triangles or rectangles. The problem with the last graph is that the curve of $y = x^2$ is rounded and we don't yet know how to work out the area of a shape with a rounded edge (except circles and ellipses). This is where integration comes in. Integration is the method used to determine the area under the curve of a function, no matter what its shape is.

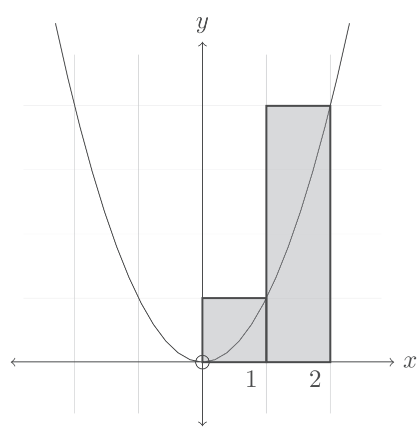
Approximate area

As a build-up to how intergration works we will start by looking at a method of finding the approximate area under a curve. Let's start off with the previous example, the one that we couldn't solve:

$y = x^2$



We start off by approximating the shaded area by using rectangles. This may seem like a silly method but just go with it for now. The idea will become more complex as we go on. By the end of this lesson you will see how well this method works.



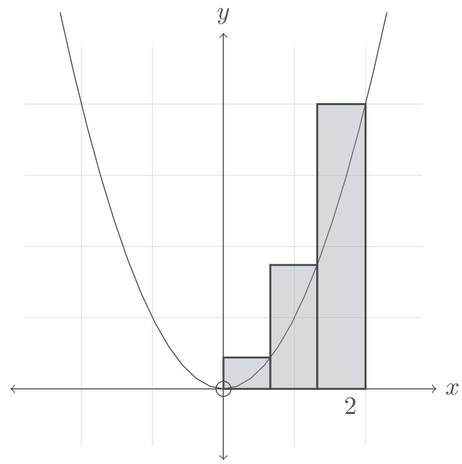
I have drawn two rectangles over the shaded area, each with the same width. Also, the upper right corner of each is touching the curve. I can then work out the area of each rectangle to get an approximate value of the area under the curve, between $x = 0$ and $x = 2$.

Question 4:

Calculate the area of this approximation.

Area of first rectangle:	Area of second rectangle:	Total area:

However, it should be clear that this is not a very good approximation. It should be clear from the diagram that the area of the rectangles is way more than the area under the curve. So how do we make our approximation better? We use more rectangles!



The area of these 3 rectangles looks like a better approximation of the shaded area, still not good, but at least better.

Question 5:

Find the value of this new approximation by calculating the area of the 3 rectangles.

Hints:

Finding the width of each rectangle: You know that their widths are equal so, to find the width for each one you have to divide the length of the x-axis interval (which is 2) by the number of rectangles.

Finding the height of each rectangle: You know that the top right corner of each rectangle touches the curve. So the height of each rectangle is equal to the function value at that point. All you have to do is plug the x-value at the bottom right corner of each rectangle into the function and that will give you the height of the rectangle.

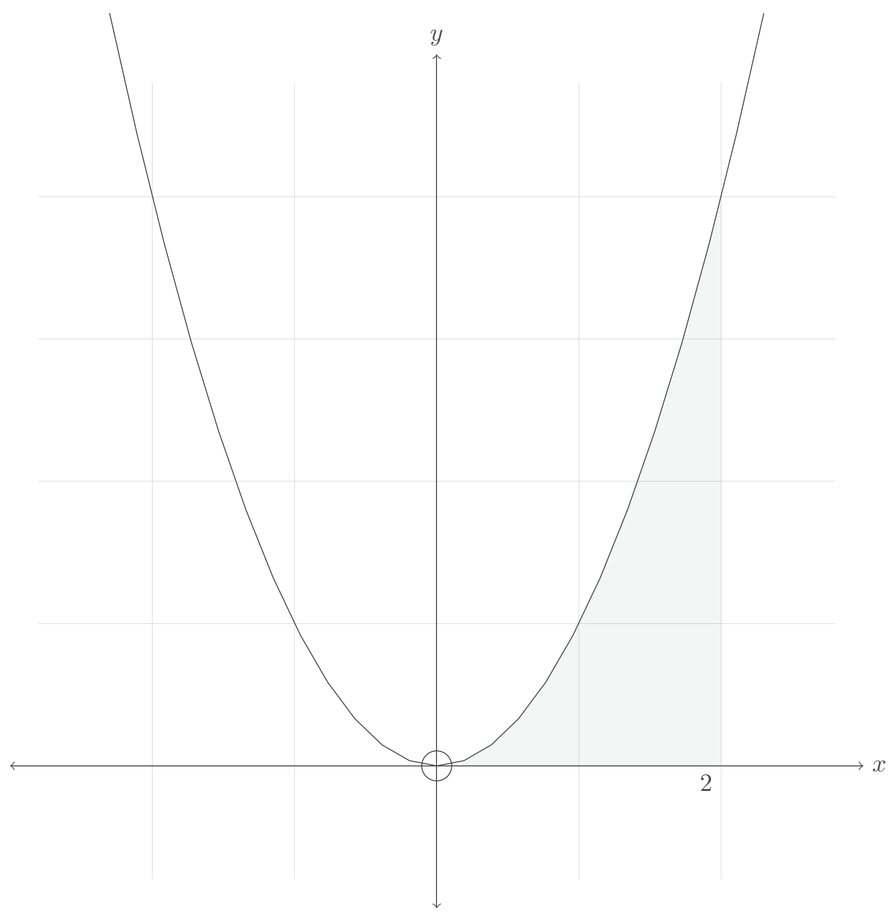
Area of first rectangle:	Area of second rectangle:	Area of third rectangle:	Total area:

Question 6:

Now that you have the idea of using rectangles to approximate the area under the function, use 5 rectangles to approximate the same area. Draw the rectangles on the graph that follows:

Area of first rectangle:	
Area of second rectangle:	
Area of third rectangle:	
Area of forth rectangle:	
Area of fifth rectangle:	
Total area:	

5 rectangle approximation

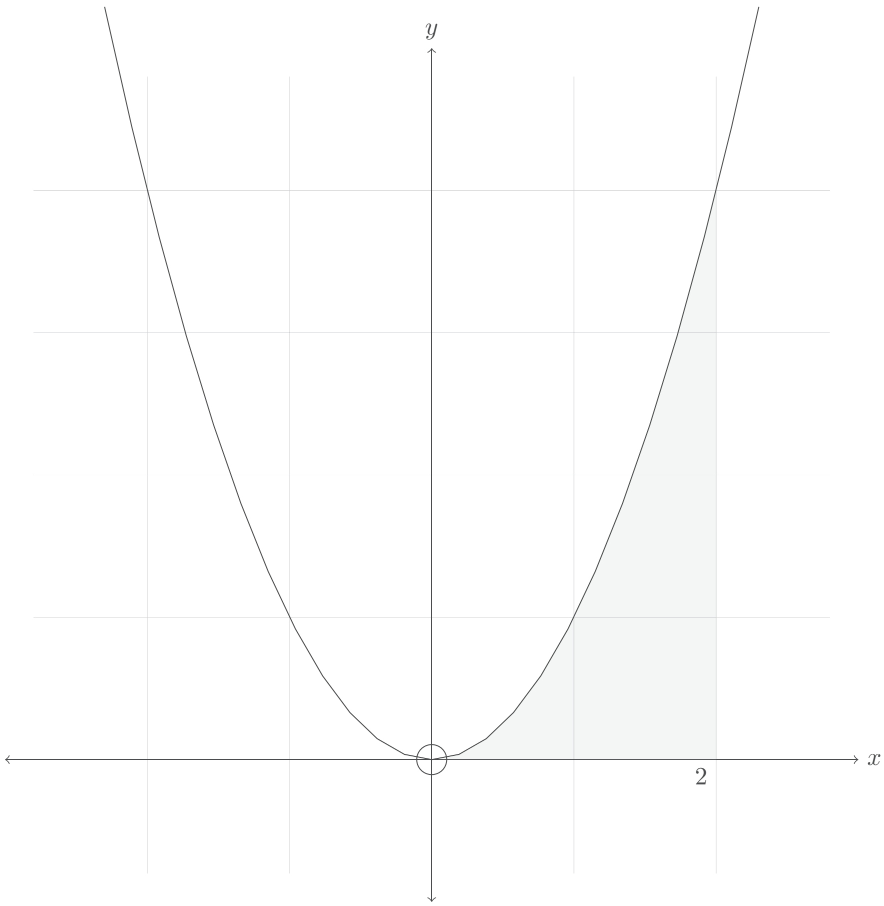


Question 7:

Use 10 rectangles to approximate the same area again. Draw the rectangles on the graph that follows: (This will take some time but doing this exercise will get you used to the very basic ideas behind integration, thereby making the work after this question easier to grasp.)

Area of first rectangle:	
Area of second rectangle:	
Area of third rectangle:	
Area of forth rectangle:	
Area of fifth rectangle:	
Area of sixth rectangle:	
Area of seventh rectangle:	
Area of eighth rectangle:	
Area of ninth rectangle:	
Area of tenth rectangle:	
Total area:	

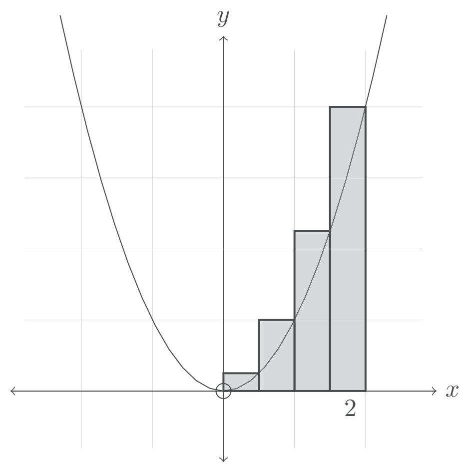
10 rectangle approximation



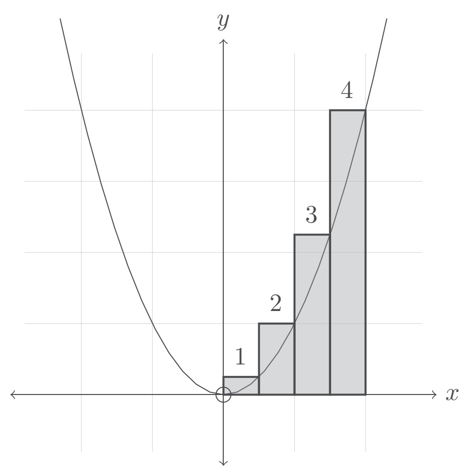
It should be clear from the diagrams above that the more rectangles you use, the closer the sum of the area of the rectangles gets to the area under the curve. In other words, the approximation of the area under the curve gets better.

A rectangle area formula

If we are asked to use rectangles to approximate the area under a specific curve we can make our lives just a little bit easier by finding a formula that gives us the area of any rectangle under that curve. We will be looking at the same example as before, this time using 4 rectangles for the approximation, to see how this works.



If we number each rectangle from left to right as follows:



Then our formula should work like this: If we want the area of rectangle 3 then all we have to do is plug the number 3 into the formula and we will get the area of rectangle 3 out. But how do we get such an awesome formula?

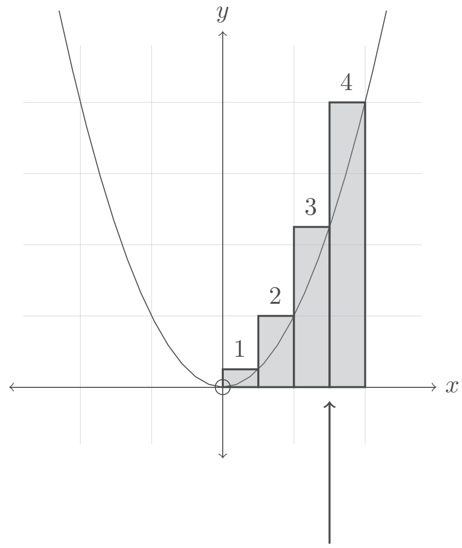
Here's how:

- First we find the width of each rectangle (remembering that all their widths are the same). To find their widths you have to divide the length of the interval on the x-axis, over which the area is being taken, by the total number of rectangles. In this example the interval length is 2 and the number of rectangles is 4. The symbol for the width of each rectangle is Δx.

$$\begin{aligned} \therefore \Delta x &= \frac{2}{4} \\ &= 0.5 \end{aligned}$$

- Now we need to find the height of each rectangle, but they all have different heights. So we need to find a formula that gives the height of any rectangle if we put that rectangles number into the formula.

We saw earlier that if we want to find the height of a rectangle then we have to plug the x-value at the bottom right corner of that rectangle into the function. This is because the function value at this value for x is equal to the height of the rectangle.



To find the hight of rectangle 3, we have to plug this x-value into the function.

But how do we find this x-value? Notice how the x-value at that point is 3 rectangle widths away from the origin. Therefore the x-value at this point must be equal to $3 \times \Delta x$. Therefore the height of rectangle 3 is equal to $f(3 \times \Delta x)$.

Because $f(x) = x^2$ and $\Delta x = 0.5$ the height of rectangle 3 is equal to $(3 \times \Delta x)^2$. In general, if we let i represent the rectangle number, we get the following formula for the height of a rectangle under a curve: $f(i \times \Delta x)$.

Now that we know how to find the height of each rectangle, using only the rectangle's number, we can multiply $f(i \times \Delta x)$ by Δx and that gives us a formula for the area of a rectangle under the function:

$$\boxed{\text{Area of rectangle } i = \Delta x \cdot f(i \times \Delta x)}$$

Question 8:

Fill in the following table for the example above:

Rectangle number:	x-value at bottom right corner:	Height:	Area:
1			
2			
3			
4			
i			

Remember that we have to add all the areas of the rectangles together to give us our approximation.

Question 9:

Calculate the total area of the approximation:

Area:

A formula for the total area of the approximation

So far we have a formula to represent the area of each rectangle, but we have to add these areas to get the area of the approximation. Can't we find a formula that represents the whole area of the approximation? Yes we can, using sum notation.

This is how we found the area of the approximation in the previous question:

$$\text{Approximate area} = 0.5 \times (1 \times 0.5)^2 + 0.5 \times (2 \times 0.5)^2 + 0.5 \times (3 \times 0.5)^2 + 0.5 \times (4 \times 0.5)^2$$

This represents the sum of the 4 rectangles. So it shouldn't be too difficult to see that this can all be written as:

$$\sum_{i=1}^{4} 0.5 \times (i \times 0.5)^2$$

The general formula for the approximate area under any function using any number of rectangles is:

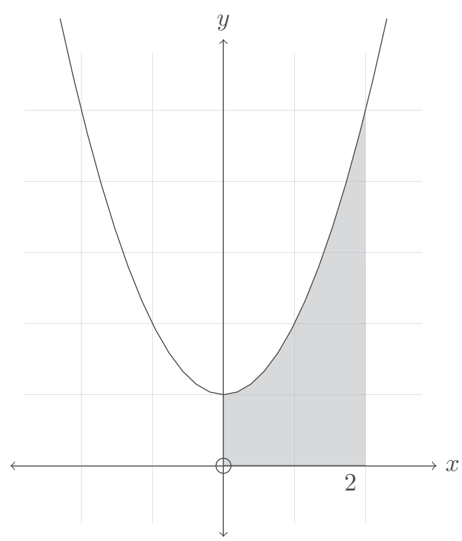
$$\boxed{\sum_{i=1}^{n} \Delta x \cdot f(i \cdot \Delta x)}$$

We need to get some practice before we move on. What I'm going to do is give you a whole lot of graphs and then ask you to write out the formula for the approximate area using the details that I give you. Here is an example before we get started:

Question:

Write out the formula for the approximate area under the curve, between $x = 0$ and $x = 2$, using 4 rectangles. Then calculate the approximate area (you will only have to do this for the first example).

$$f(x) = x^2 + 1$$



Answer:

Find Δx:

$$\begin{aligned}\Delta x &= \frac{(2-0)}{4} \\ &= \frac{2}{4} \\ &= 0.5\end{aligned}$$

Create the formula:

$$\begin{aligned}\text{Area approximation} &= \sum_{i=1}^{4} 0.5 \times f(i \times 0.5) \\ &= \sum_{i=1}^{4} 0.5 \times [(i \times 0.5)^2 + 1]\end{aligned}$$

Once you have found the formula you may be asked to use the formula to calculate the area of the approximation under the function.

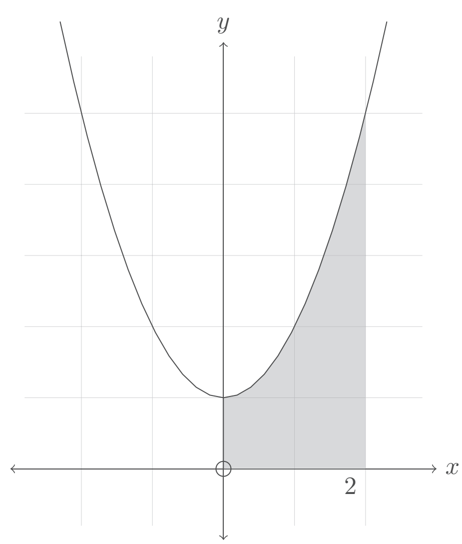
$$
\begin{aligned}
&\sum_{i=1}^{4} 0.5 \times [(i \times 0.5)^2 + 1] \\
=&\sum_{i=1}^{4} 0.5 \times (i^2 \times 0.25 + 1) \\
=&\sum_{i=1}^{4} (i^2 \times 0.125 + 0.5) \\
=&(1^2 \times 0.125 + 0.5) + (2^2 \times 0.125 + 0.5) + (3^2 \times 0.125 + 0.5) + (4^2 \times 0.125 + 0.5) \\
=&0.625 + 1 + 1.625 + 2.5 \\
=&5.75
\end{aligned}
$$

Study this example very carefully and make sure you understand every step. Then try the following questions:

Question 10:

Write out the formula for the approximate area under the curve, between $x = 0$ and $x = 2$, using 8 rectangles. Then calculate the approximate area.

$$f(x) = x^2 + 1$$

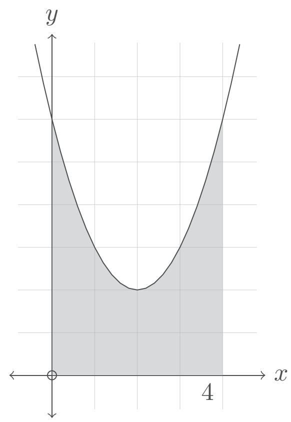


Formula:	
Approximate area:	

Question 11:

Write out the formula for the approximate area under the curve, between $x = 0$ and $x = 4$, using 4 rectangles.

$$f(x) = x^2 - 4x + 6$$

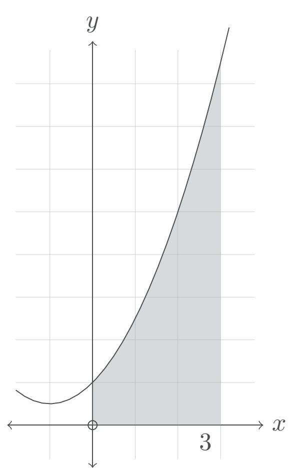


Formula:	

Question 12:

Write out the formula for the approximate area under the curve, between $x = 0$ and $x = 3$, using 9 rectangles.

$$f(x) = \frac{1}{2}x^2 + x + 1$$



Formula:	

Simplifying the formula

What we have looked at so far is how to use rectangles to approximate the area under a curve and how to represent the area of the rectangle approximation in sum notation. The problem is, if we want to calculate the areas of the rectangles then we have to calculate each one of their areas separately. This can take a lot of time and it gets boring very quickly. Wouldn't it be better if we could calculate the area of ALL the rectangles in one go? We will now see that, once we have written out rectangle area approximation in sum notation, we can.

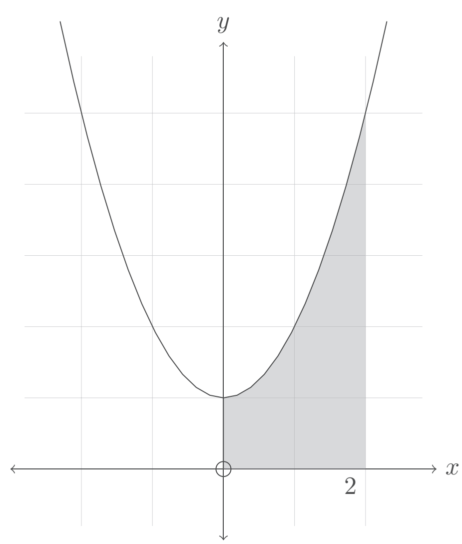
Remember these formulae?:

$$\boxed{\begin{aligned} \sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\ \sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{i=1}^{n} i^3 &= \frac{n^2(n+1)^2}{4} \end{aligned}}$$

If not, look back over your school notes. These formulae give us a quick way of adding all the terms in these particular types of series by simply plugging in n.

As an example of how to use these summation formulae to work out the total area of the rectangle approximation, we will use them to calculate the approximate area of:

$$f(x) = x^2 + 1$$



This was the worked example that we looked at just before question 10. If you look back at that example, you will see how we had to calculate the area of each rectangle separately. Now we will go through the new method of working out the approximate area using the summation formulae.

This is its rectangle area approximation in sum notation:

$$\sum_{i=1}^{4} 0.5 \times [(i \times 0.5)^2 + 1]$$

Once we have written the rectangle area approximation in sum notation, we can begin using the new method.

$$\begin{aligned}
&\sum_{i=1}^{4} 0.5 \times [(i \times 0.5)^2 + 1] \\
&= \sum_{i=1}^{4} (0.5 \times (i \times 0.5)^2 + 0.5) \\
&= \sum_{i=1}^{4} (0.5 \times (i \times 0.5)^2) + \sum_{i=1}^{4} 0.5 \\
&= \sum_{i=1}^{4} (0.125 \times i^2) + 2 \\
&= 0.125 \sum_{i=1}^{4} i^2 + 2
\end{aligned}$$

We know that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ and we have $n = 4$ so we get:

$$\begin{aligned}
&= 0.125 \times \frac{4(4+1)(8+1)}{6} + 2 \\
&= 5.75
\end{aligned}$$

If you look back over the example just before question 10, you will see that we get the same answer. This method may not seem that great when compared to the previous method of working out the area of each rectangle separately, but what if we wanted to use 40 rectangles to approximate this area? We don't want to calculate the areas of 40 rectangles separately. On the other hand, if we used this new method to find the sum of the areas of 40 rectangles, it would require no more working than what we have just done for calculating the sum of the areas of 4 rectangles. All you would have to do is replace $n = 4$ with $n = 40$. We also know that the more rectangles we use, the better our approximation. This method allows us to calculate very good approximations by doing the same amount of work.

Question 13:

Use the Summation formulae method to calculate the approximate area, under the curve of x^2, using the given number of rectangles: (You do not have to draw the approximations for these examples.)

	Rectangle approximation in sum notation:	Calculation of approximation:
Find the approximate area under x^2, between $x = 0$ and $x = 2$, using 10 rectangles.		
Find the approximate area under x^2, between $x = 0$ and $x = 2$, using 20 rectangles.		
Find the approximate area under x^2, between $x = 0$ and $x = 2$, using 50 rectangles.		

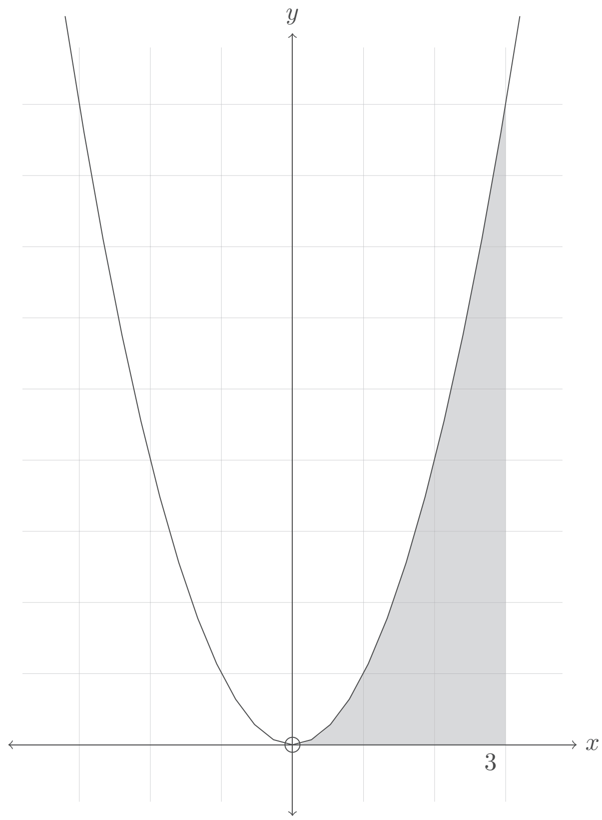
Now that we are able to use a lot of rectangles for our approximations our calculations of the approximations are much closer to the actual value of the area. But being closer does not make them equal to the exact area. If we want the exact area we will have to go one step further.

Finding the EXACT area under the curve

If we want to calculate the exact area under a curve, neither millions nor billions of rectangles will work. As long as we are using some number of rectangles, our area will always be an approximation. If we want to find the EXACT area under a curve, we need to use an infinite number of rectangles! If we want to do that... we will need to use limits. Here's how:

We will use x^2 as our function and we will attempt to calculate the exact area under the curve, between $x = 0$ and $x = 3$.

This is the area we are looking for:



We start out by writing the summation notation for our approximation, using n number of rectangles (n could be any integer). Previously, we would have chosen a particular number of rectangles but now we are just going to use n, to keep things general.

$$\sum_{i=1}^{n}\left(\frac{3}{n}\times\left(i\times\frac{3}{n}\right)^{2}\right)$$

Remember: $\frac{3}{n}$ is the width of each rectangle.

But, as long as n is an integer, $\sum_{i=1}^{n}\left(\frac{3}{n}\times\left(i\times\frac{3}{n}\right)^{2}\right)$ will only represent an approximation of the shaded area above. If we want the exact area we have to let n approach infinity.

$$\text{Exact area} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3}{n}\times\left(i\times\frac{3}{n}\right)^{2}\right)$$

The next step is to use our summation formulae to simplify $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3}{n}\times\left(i\times\frac{3}{n}\right)^{2}\right)$, like we did in the previous example. Think of this as surgery. Make sure you read through it over and over again, until you understand every step.

$$
\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3}{n}\times\left(i\times\frac{3}{n}\right)^2\right)\\
=&\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3}{n}\times i^2\times\frac{9}{n^2}\right)\\
=&\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{27}{n^3}\times i^2\right)\\
=&\lim_{n\to\infty}\frac{27}{n^3}\sum_{i=1}^{n}i^2\\
=&\lim_{n\to\infty}\left(\frac{27}{n^3}\times\frac{n(n+1)(2n+1)}{6}\right)\\
=&\lim_{n\to\infty}\frac{27(n+1)(2n+1)}{6n^2}\\
=&\lim_{n\to\infty}\frac{54n^2+81n+27}{6n^2}\\
=&\lim_{n\to\infty}\left(\frac{54}{6}+\frac{81}{6n}+\frac{27}{6n^2}\right)\\
=&\frac{54}{6}\\
=&9
\end{aligned}
$$

Therefore 9 is the EXACT value of the shaded area above.

In general, the formula for the exact area under a curve is:

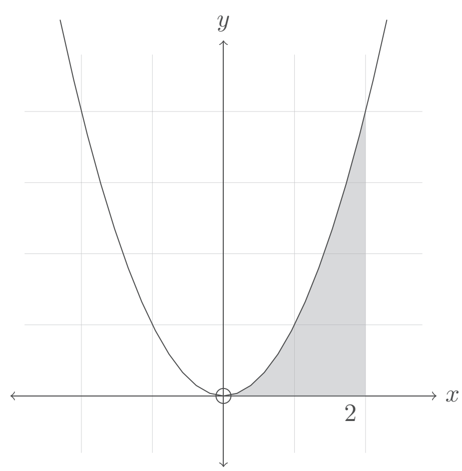
$$
\boxed{\lim_{n\to\infty}\sum_{i=1}^{n}\Delta x\cdot f(i\cdot\Delta x)}
$$

This method is called integration by first principles and this is the method that we have been building up to.

Question 14:

To start off with some practice, let's look back over the whole lesson and use the approximate values that we have already calculated, for the graph below, to fill in the table below.

$y = x^2$



Number of rectangles used in approximation:	Area of approximation:
2	
3	
5	
10	
20	
50	

Question 15:

Use first principles to find the exact shaded area for the graph in question 14: (looking back over the worked example, just before question 14, will help you if you get stuck)

Exact area =

Integration by first principles gives us exactly what we need to solve areas under curves, so we could stop here. However, there is a method that makes integration much easier and quicker. We will spend the last lesson looking at this method.

Solutions

Solution 1:

$$\begin{aligned}\text{Area} &= l \times b\\ &= 3 \times 4\\ &= 12\end{aligned}$$

Solution 2:

$$\begin{aligned}\text{Area} &= \frac{1}{2} \times b \times h\\ &= \frac{1}{2} \times 2 \times 4\\ &= 4\end{aligned}$$

Solution 3:

$$\text{Area} = (4 \times 2) + \left(\frac{1}{2} \times 4 \times 4\right) = 16$$

Solution 4:

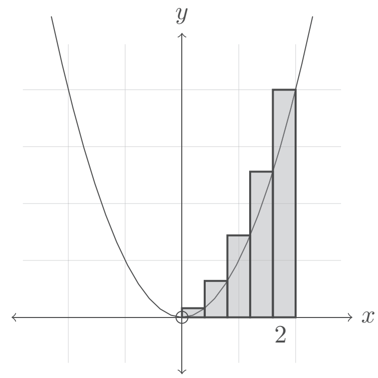
Area of first rectangle:	Area of second rectangle:	Total area:
$1 \times 1 = 1$	$1 \times 4 = 4$	$1 + 4 = 5$

Solution 5:

Area of first rectangle:	Area of second rectangle:	Area of third rectangle:	Total area:
$\text{Base} = \frac{2}{3}$ $\text{Height} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$ $\text{Area} = \frac{2}{3} \times \frac{4}{9} = \frac{8}{27}$	$\text{Base} = \frac{2}{3}$ $\text{Height} = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$ $\text{Area} = \frac{2}{3} \times \frac{16}{9} = \frac{32}{27}$	$\text{Base} = \frac{2}{3}$ $\text{Height} = (2)^2 = 4$ $\text{Area} = \frac{2}{3} \times 4 = \frac{8}{3}$	4.15

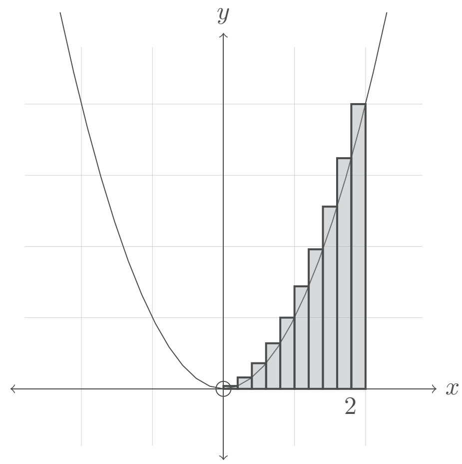
Solution 6:

Area of first rectangle:	$\text{Base} = \frac{2}{5}$	$\text{Height} = \left(\frac{2}{5}\right)^2 = \frac{4}{25}$	$\text{Area} = \frac{2}{5} \times \frac{4}{25} = \frac{8}{125}$
Area of second rectangle:	$\text{Base} = \frac{2}{5}$	$\text{Height} = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$	$\text{Area} = \frac{2}{5} \times \frac{16}{25} = \frac{32}{125}$
Area of third rectangle:	$\text{Base} = \frac{2}{5}$	$\text{Height} = \left(\frac{6}{5}\right)^2 = \frac{36}{25}$	$\text{Area} = \frac{2}{5} \times \frac{36}{25} = \frac{72}{125}$
Area of forth rectangle:	$\text{Base} = \frac{2}{5}$	$\text{Height} = \left(\frac{8}{5}\right)^2 = \frac{64}{25}$	$\text{Area} = \frac{2}{5} \times \frac{64}{25} = \frac{128}{125}$
Area of fifth rectangle:	$\text{Base} = \frac{2}{5}$	$\text{Height} = (2)^2 = 4$	$\text{Area} = \frac{2}{5} \times 4 = \frac{8}{5}$
Total area:	$\frac{8}{125} + \frac{32}{125} + \frac{72}{125} + \frac{128}{125} + \frac{8}{5} = \frac{440}{125} = 3.52$		



Solution 7:

Area of first rectangle:	Base $= \frac{2}{10}$ Height $= \left(\frac{2}{10}\right)^2 = \frac{4}{100}$ Area $= \frac{2}{10} \times \frac{4}{100} = \frac{8}{1000}$
Area of second rectangle:	Base $= \frac{2}{10}$ Height $= \left(\frac{4}{10}\right)^2 = \frac{16}{100}$ Area $= \frac{2}{10} \times \frac{16}{100} = \frac{32}{1000}$
Area of third rectangle:	Base $= \frac{2}{10}$ Height $= \left(\frac{6}{10}\right)^2 = \frac{36}{100}$ Area $= \frac{2}{10} \times \frac{36}{100} = \frac{72}{1000}$
Area of forth rectangle:	Base $= \frac{2}{10}$ Height $= \left(\frac{8}{10}\right)^2 = \frac{64}{100}$ Area $= \frac{2}{10} \times \frac{64}{100} = \frac{128}{1000}$
Area of fifth rectangle:	Base $= \frac{2}{10}$ Height $= (1)^2 = 1$ Area $= \frac{2}{10} \times 1 = \frac{2}{10}$
Area of sixth rectangle:	Base $= \frac{2}{10}$ Height $= \left(\frac{12}{10}\right)^2 = \frac{144}{100}$ Area $= \frac{2}{10} \times \frac{144}{100} = \frac{288}{1000}$
Area of seventh rectangle:	Base $= \frac{2}{10}$ Height $= \left(\frac{14}{10}\right)^2 = \frac{196}{100}$ Area $= \frac{2}{10} \times \frac{196}{100} = \frac{392}{1000}$
Area of eighth rectangle:	Base $= \frac{2}{10}$ Height $= \left(\frac{16}{10}\right)^2 = \frac{256}{100}$ Area $= \frac{2}{10} \times \frac{256}{100} = \frac{512}{1000}$
Area of ninth rectangle:	Base $= \frac{2}{10}$ Height $= \left(\frac{18}{10}\right)^2 = \frac{324}{100}$ Area $= \frac{2}{10} \times \frac{324}{100} = \frac{648}{1000}$
Area of tenth rectangle:	Base $= \frac{2}{10}$ Height $= (2)^2 = 4$ Area $= \frac{2}{10} \times 4 = \frac{8}{10}$
Total area:	$\frac{8}{1000} + \frac{32}{1000} + \frac{72}{1000} + \frac{128}{1000} + \frac{2}{10} + \frac{288}{1000} + \frac{392}{1000} + \frac{512}{1000} + \frac{648}{1000} + \frac{8}{10} = \frac{3080}{1000} = 3.08$



Solution 8:

Rectangle number:	x-value at bottom right corner:	Height:	Area:
1	0.5	0.25	$0.5 \times 0.25 = 0.125$
2	1	1	$1 \times 1 = 1$
3	1.5	2.25	$1.5 \times 2.25 = 3.38$
4	2	4	$2 \times 4 = 8$
i	$0.5 \times i$	$(0.5 \times i)^2$	$(0.5 \times i) \times (0.5 \times i)^2$

Solution 9:

0.125+1+3.38+8=12.5

Solution 10:

Formula:	$\Delta x = \frac{(2-0)}{8} \quad \sum_{i=1}^{8} 0.25 \times f(i \times 0.25)$ $= \frac{2}{8} \quad = \sum_{i=1}^{8} 0.25 \times [(i \times 0.25)^2 + 1]$ $= 0.25 \quad = \sum_{i=1}^{8} (0.016i^2 + 0.25)$
Approximate area:	$(0.016 \times 1^2 + 0.25) + (0.016 \times 2^2 + 0.25) + (0.016 \times 3^2 + 0.25) +$ $(0.016 \times 4^2 + 0.25) + (0.016 \times 5^2 + 0.25) + (0.016 \times 6^2 + 0.25) +$ $(0.016 \times 7^2 + 0.25) + (0.016 \times 8^2 + 0.25) = 5.26$

Solution 11:

Formula:	$\Delta x = \frac{(4-0)}{4} \quad \sum_{i=1}^{4} 1 \times [(1 \times i)^2 - 4(1 \times i) + 6]$ $= 1 \quad = \sum_{i=1}^{4} (i^2 - 4i + 6)$

Solution 12:

Formula:	$\Delta x = \dfrac{(3-0)}{9} = \dfrac{1}{3}$ $\sum_{i=1}^{9} \dfrac{1}{3} \times \left[\dfrac{1}{2}\left(\dfrac{1}{3}i\right)^2 + \left(\dfrac{1}{3}i\right) + 1\right] = \sum_{i=1}^{9}\left(\dfrac{1}{54}i^2 + \dfrac{1}{9}i + \dfrac{1}{3}\right)$

Solution 13:

	Rectangle approximation in sum notation:	Calculation of approximation:
Find the approximate area under x^2, between $x = 0$ and $x = 2$, using 10 rectangles.	$\sum_{i=1}^{10} \frac{2}{10} \times \left(i \times \frac{2}{10}\right)^2$ $\sum_{i=1}^{10} \frac{8}{1000} \times i^2$	$\sum_{i=1}^{10} \frac{8}{1000} \times i^2$ $= \frac{8}{1000} \times \frac{10(10+1)(20+1)}{6}$ $=3.08$
Find the approximate area under x^2, between $x = 0$ and $x = 2$, using 20 rectangles.	$\sum_{i=1}^{20} \frac{2}{20} \times \left(i \times \frac{2}{20}\right)^2$ $\sum_{i=1}^{20} \frac{8}{8000} \times i^2$	$\sum_{i=1}^{20} \frac{8}{8000} \times i^2$ $= \frac{8}{8000} \times \frac{20(20+1)(40+1)}{6}$ $=2.87$
Find the approximate area under x^2, between $x = 0$ and $x = 2$, using 50 rectangles.	$\sum_{i=1}^{50} \frac{2}{50} \times \left(i \times \frac{2}{50}\right)^2$ $\sum_{i=1}^{50} \frac{8}{125000} \times i^2$	$\sum_{i=1}^{50} \frac{8}{125000} \times i^2$ $= \frac{8}{125000} \times \frac{50(50+1)(100+1)}{6}$ $=2.75$

Solution 14:

Number of rectangles used in approximation:	Area of approximation:
2	5
3	4.15
5	3.52
10	3.08
20	2.87
50	2.75

Solution 15:

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2}{n}\times\left(i\times\frac{2}{n}\right)^2\right)\\
=&\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{8}{n^3}i^2\right)\\
=&\lim_{n\to\infty}\left(\frac{8}{n^3}\times\frac{n(n+1)(2n+1)}{6}\right)\\
=&\lim_{n\to\infty}\frac{16n^3+24n^2+8n}{6n^3}\\
=&\lim_{n\to\infty}\left(\frac{16+\frac{24}{n}+\frac{8}{n^2}}{6}\right)\\
=&\frac{16}{6}
\end{aligned}$$

$\therefore$Exact area = 2.6666666...

Integration

Lesson 2: Integration by anti-differentiation

In life there is the easy way and the difficult way of doing something. I am pleased to let you know that you have already learnt the difficult way of integrating. In this final lesson we will look at the easy way. We will look at how to integrate using anti-differentiation.

Anti-differentiation

Let's put integration aside for a little bit and focus on what anti-differentiation is all about. You know what differentiation is; if I ask you to differentiate x^2, you can do it very quickly, using the polynomial rule:

Question 1:

$$\begin{array}{rl} \text{If} & f(x) = x^2 \\ \text{Then} & f'(x) = \underline{\hspace{3cm}}? \end{array}$$

Now, if I ask you to anti-differentiate x^2 then I am asking you to give me the function that, when differentiated, equals x^2. In other words:

Question 2:

$$\begin{array}{rl} \text{If} & g(x) = \underline{\hspace{3cm}}? \\ \text{Then} & g'(x) = x^2 \end{array}$$

So now we have to work backwards. We have to use the polynomial rule in reverse.

The polynomial rule says:

- Multiply the coefficient by the exponent.
- Subtract 1 from the exponent.

The polynomial rule in reverse goes like this:

- Add one to the exponent.
- Divide the coefficient by the exponent.

So we find the anti-derivative of x^2 like this:

- $x^{2+1} = x^3$
- $(1/3)x^3 = \frac{1}{3}x^3$ (Remember that the coefficient of x^3 is 1.)

To make sure that $\frac{1}{3}x^3$ is the anti-derivative of x^2, find the derivative $\frac{1}{3}x^3$:

Question 3:

$$\begin{array}{rl} \text{If} & f(x) = \frac{1}{3}x^3 \\ \text{Then} & f'(x) = \underline{\hspace{3cm}}? \end{array}$$

The lost constant

Now that we have the basic idea behind anti-differentiating polynomials, there is just one small detail that needs our attention before we can start looking at the new method of integration. I will illustrate this idea with a question:

Question 4:

$$\begin{array}{rl} \text{If} & f(x) = 1 \\ \text{Then} & f'(x) = \underline{\hspace{2cm}}? \end{array} \qquad \begin{array}{rl} \text{If} & g(x) = 5 \\ \text{Then} & g'(x) = \underline{\hspace{2cm}}? \end{array} \qquad \begin{array}{rl} \text{If} & h(x) = c \quad c \in \mathbb{R} \\ \text{Then} & h'(x) = \underline{\hspace{2cm}}? \end{array}$$

This is a property of differentiation that we already know about. The derivative of any constant is 0. This is not a problem if we want to differentiate a function...

If we differentiate $x^2 + 3$
we get $2x$

If we differentiate $x^2 + 7$
we get $2x$

...and so on. So what is the anti-derivative of $2x$? Before, we would have just said x^2. But, as we have seen, x^2 could have had a constant of any value added to it that would have been lost by differentiating it. So if we anti-differentiate x^2 we say:

$$\begin{array}{rl} \text{If} & f'(x) = 2x \\ \text{Then} & f(x) = x^2 + c \qquad \text{where } c \in \mathbb{R} \end{array}$$

We let c represent all possibilities for the constant and by writing $c \in \mathbb{R}$, we are saying that the constant could be any real number.

Here are some more examples to look at. Study these examples carefully and understand how each term is calculated.

$$\begin{array}{rl} \text{If} & g'(x) = 4x^3 \\ \text{Then} & g(x) = x^4 + c \end{array}$$

$$\begin{array}{rl} \text{If} & f'(x) = 9x^2 \\ \text{Then} & f(x) = 3x^3 + c \end{array}$$

$$\begin{array}{rl} \text{If} & h'(x) = 3x^2 + 2x + 1 \\ \text{Then} & h(x) = x^3 + x^2 + x + c \end{array}$$

Question 5:

Anti-differentiate the following functions: (Hint: you can always double check your answer by differentiating it to see if you get the original function.)

1	
x	
$x+1$	
$2x+3$	
0	
$12x^2-5$	
x^3	
x^4	
x^{1000}	
$4x^3-6x^2+10x-8$	

What we have covered so far is all you need to know about anti-differentiation, for now. We have only looked at how to anti-differentiate polynomials, however, there are all sorts of other functions that require different techniques of anti-differentiation. However, we won't worry about them here.

Back to integration

In the previous lesson we saw that the exact area under a function can be found using this formula:

$$\boxed{\lim_{n\to\infty}\sum_{i=1}^{n}\Delta x.f(i.\Delta x)}$$

So, looking at the example at the end of the last lesson, if we wanted to work out the exact area under the function x^2, between $x=0$ and $x=3$, we would write our formula out like this:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(i.\frac{3}{n}\right)^2$$

We would then have to use the first principles method to work it out. But, with anti-differentiation, those days are over. There is a much quicker method and you will either be very happy or very upset to see how quick it is. This is what the new formula looks like:

$$\boxed{\int_a^b f(x)dx}$$

Don't panic! I will show you what it means and how to use it. However, I will not show you why it works. You will just have to trust me and wait for first year to see why it works. For now, here's how it works:

I will use the example from the end of the last lesson to demonstrate how to use this formula. For that example we wanted to find the exact area under x^2 , between $x = 0$ and $x = 3$. This is how we substitute this information into the new formula:

$$\int_0^3 x^2 dx$$

What do we do now? We have to anti-differentiate x^2 and put it in square brackets:

$$\left[\frac{1}{3}x^3 + c\right]_0^3$$

And what are the 3 and 0 doing on the outside of the square brackets? We substitute 3 into the anti-derivative, then we substitute 0 into the anti-derivative and then we subtract the second result from the first:

$$\left[\frac{1}{3}(3)^3 + c\right] - \left[\frac{1}{3}(0)^3 + c\right]$$

And then solve:

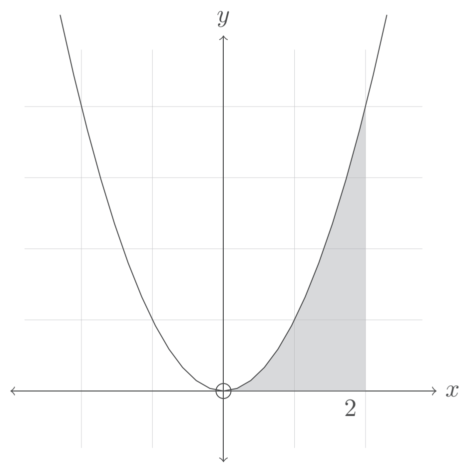
$$\begin{aligned} &= \frac{27}{3} + c - \frac{0}{3} - c \\ &= 9 - 0 \\ &= 9 \end{aligned}$$

You may be wondering why the $\int$ and the dx are in the formula. That's not very important for now. Think of them as the notation that tells us to use this method.

Question 6:

Find the shaded area in our favorite example, using the anti-differentiation method, and compare your answer to the result that you got at the end of the previous lesson.

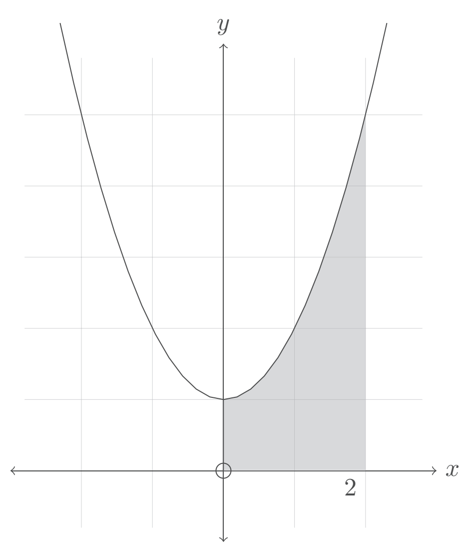
$f(x) = x^2$



Question 7:

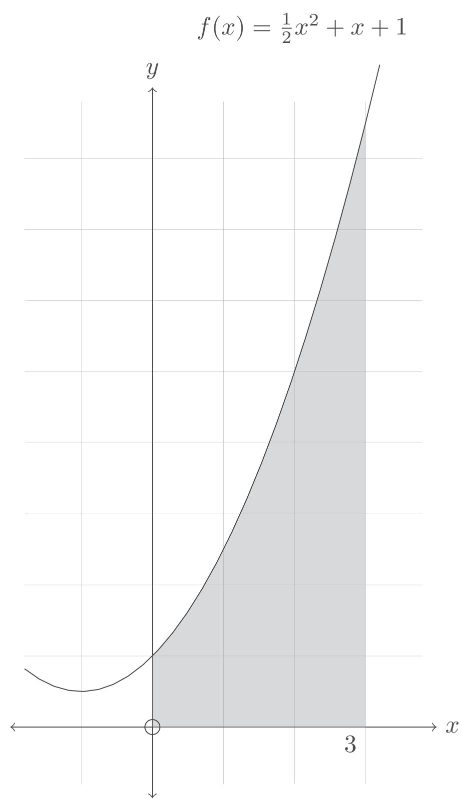
Find the following shaded area:

$f(x) = x^2 + 1$



Question 8:

Find the following shaded area:



Intervals

You may have noticed by now that every interval that we have integrated over has started at 0 and ended at some positive integer on the x-axis. The only reason for this is that it makes understanding integration a bit easier. In general we can integrate over any interval.

If we want to find the area under the function x^2 between $x = 3$ and $x = 5$, all we do is this:

$$\int_3^5 x^2 dx$$

We then use the method exactly as before.

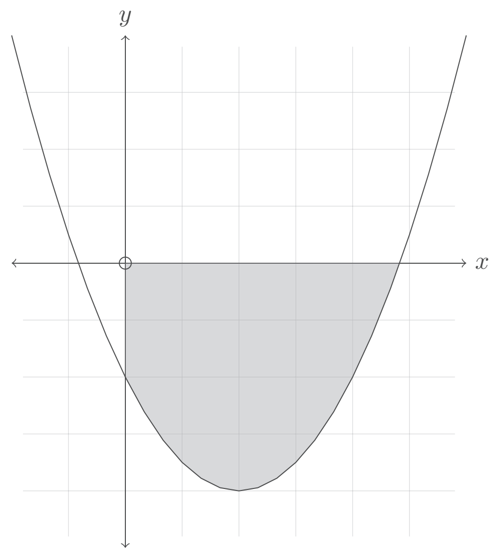
Question 9:

Solve the above integral:

$$\int_3^5 x^2 dx =$$

Negative area

This is something we never hear about but it can happen when integrating. If we calculate the area between the function and the x-axis and that area is below the x-axis, it will be calculated as a negative area. Example:



If we used integration to find this area, our result would be negative.

We have had a very brief look at integration but these few concepts form the very foundation of what turns out to be one of the largest sections of first year mathematics. Integrating polynomials is relatively easy and it is only when you come face to face with all the other types of functions that you will see how interesting integration can get. To be able to integrate more complicated functions will require more from you than simply learning a whole bunch of methods. It will require you to be creative. You will have to learn to approach integration problems with an open but analytical mind. As you progress further with integration you will find that it is as much an art as it is a science.

Solutions

Solution 1:

$$\begin{aligned} \text{If} \qquad & f(x) = x^2 \\ \text{Then} \qquad & f'(x) = 2x \end{aligned}$$

Solution 2:

$$\begin{aligned} \text{If} \qquad & g(x) = \frac{1}{3}x^3 \\ \text{Then} \qquad & g'(x) = x^2 \end{aligned}$$

Solution 3:

$$\begin{aligned} \text{If} \qquad & f(x) = \frac{1}{3}x^3 \\ \text{Then} \qquad & f'(x) = 2x \end{aligned}$$

Solution 4:

$$\begin{aligned} \text{If} \quad & f(x) = 1 & \text{If} \quad & g(x) = 5 & \text{If} \quad & h(x) = k \quad k \in \mathbb{R} \\ \text{Then} \quad & f'(x) = 0 & \text{Then} \quad & g'(x) = 0 & \text{Then} \quad & h'(x) = 0 \end{aligned}$$

Solution 5:

1	$x + c$
x	$\frac{1}{2}x^2 + c$
$x + 1$	$\frac{1}{2}x^2 + x + c$
$2x + 3$	$x^2 + 3x + c$
0	c
$12x^2 - 5$	$4x^3 - 5x + c$
x^3	$\frac{1}{4}x^4 + c$
x^4	$\frac{1}{5}x^5 + c$
x^{1000}	$\frac{1}{1001}x^{1001} + c$
$4x^3 - 6x^2 + 10x - 8$	$x^4 - 2x^3 + 5x^2 - 8x + c$

Solution 6:

$$
\begin{aligned}
&\int_0^2 x^2 dx \\
&= \left[\frac{1}{3}x^3 + c\right]_0^2 \\
&= \left[\frac{1}{3}(2)^3 + c\right] - \left[\frac{1}{3}(0)^3 + c\right] \\
&= \frac{8}{3} + c - \frac{0}{3} - c \\
&= \frac{8}{3} \\
&= 2.666666...
\end{aligned}
$$

Solution 7:

$$\begin{aligned}
&\int_0^2 (x^2+1)dx \\
=&\left[\frac{1}{3}x^3+x+c\right]_0^2 \\
=&\left[\frac{1}{3}(2)^3+(2)+c\right]-\left[\frac{1}{3}(0)^3+(0)+c\right] \\
=&\frac{8}{3}+2+c-\frac{0}{3}-0-c \\
=&\frac{14}{3} \\
=&4.666666...
\end{aligned}$$

Solution 8:

$$\begin{aligned}
&\int_0^3 \left(\frac{1}{2}x^2+x+1\right)dx \\
=&\left[\frac{1}{6}x^3+\frac{1}{2}x^2+x+c\right]_0^3 \\
=&\left[\frac{1}{6}(3)^3+\frac{1}{2}(3)^2+(3)+c\right]-\left[\frac{1}{6}(0)^3+\frac{1}{2}(0)^2+(0)+c\right] \\
=&\frac{27}{6}+\frac{9}{2}+3+c-\frac{0}{6}-\frac{0}{2}-0-c \\
=&12
\end{aligned}$$

Solution 9:

$$\begin{aligned}
&\int_3^5 x^2dx \\
=&\left[\frac{1}{3}x^3+c\right]_3^5 \\
=&\left[\frac{1}{3}(5)^3+c\right]-\left[\frac{1}{3}(3)^3+c\right] \\
=&\frac{125}{3}+c-\frac{27}{3}-c \\
=&\frac{98}{3} \\
=&32.666666...
\end{aligned}$$

Farewell

We have been through a lot together but now I must say good bye. It is time for you to enter the world of university mathematics on your own.

Good luck! I mean it. University is a challenging environment. You will be given a level of freedom beyond that which you have ever experienced in school. How well you do is up to you. You will have to set your own study routine and motivate yourself to keep up with the work. There is no detention, no teacher to keep you on your toes. It's all up to you.

With such freedom comes the inclination to 'relax today and work tomorrow'. Don't let this disease infect your university life. Rather take the 'if it can be done today then it must be done today' approach. You will get much further.

Another cause of failure among students is distraction. With your new found freedom you may think, "this is my life now, I can do what I like". True, you can do what you like. You can bunk class, party all night and spend all day watching TV. Don't let this way of life become a habit. Save the partying for after the tests and exams. This way you will have something to celebrate without the pressure of work and studying to put the brakes on your evening.

Success is about balance, find the right balance between work and relaxing and you will reap the rewards.

I wish you all the best for your studies.

Chris

How it began

I started teaching school mathematics to learners in 2002, when I was in grade 11, at Fish Hoek Senior High, Cape Town, South Africa. In 2006 I began a science degree and it was during this time that my private tutoring experience continued to grow. In 2009, during the final year of my degree, I was asked to introduce the learners of COSAT (Centre of Science and Technology), then based at the False Bay College, Cape of Good Hope Campus in Khayelitsha, to university mathematics.

What makes COSAT a special place is that it draws in some of the brightest learners of the area, all of whom have the potential to become first class university graduates. However, it takes more than potential to attain such a desirable goal.

Many equally talented learners have made it to first year but have had their student lives cut short due to inadequate preparation. An unnecessary waste! By the end of 2009 I had realised that a solid foundation to university mathematics was essential to all grade 12 learners intending to progress that far. To be able to reach all these learners, I had to turn my lessons into a workbook.

At the end of 2009 I graduated with a BSc in mathematics and was now free to devote all my time to introducing the next class of grade 12's to university mathematics. It was during 2010 that *Unimaths Intro* was born. As each lesson was written, it was taught to the grade 12's of that year. Their feedback was then used to adjust the lessons to their exact needs.

Following these developmental years, *Unimaths Intro* has grown into the workbook that you now see before you.